Chapter # 12

ARITHMETIC REPETITION IN THE TEACHING AND LEARNING OF ALGEBRA AND FUNCTIONS IN UPPER SECONDARY SCHOOL

Reality status and development directions

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ABSTRACT
This study examines students' learning of arithmetic–algebra content, focusing on the transition from prior arithmetic knowledge to algebra in relation to how repetition is organized in teaching. The focus was on repetition of arithmetic content, as well as arithmetic repetition–concept shifts into new learning contexts, including algebraic expressions, equations, and the concept of function. In this study, teaching and students' subject-specific learning were analyzed according to the methodological design of Freudenthal's didactical phenomenology, with a focus on the relationship between the teaching of arithmetic and algebra, and Bernstein's conceptualization of the repetition–without–repetition approach, which concerns how prior knowledge should be systematized and integrated with new knowledge. The study illustrates that students' prior knowledge is used to a very limited extent in teaching and in a one-sided manner. The teaching of algebra is seen as an isolated act and has no conceptual connection to students' arithmetic knowledge. The following chapter sets out recommendations to address this problem and outlines an alternative approach to the teaching and learning of algebra and functions that incorporates repetition. The study highlights the importance of developing an alternative approach to "get through" learning-oriented repetition and teaching, grounded in a psychological perspective.

Keywords: arithmetic repetition, systematic repetition, learning-based teaching, algebra learning, psychology, students' learning, Freudenthal's didactical phenomenology, Bernstein's repetition-without-repetition approach.

1. INTRODUCTION: CONCEPTUAL RESEARCH CONTEXT

Students' knowledge transition from the final year of primary school to secondary school is perceived as problematic and challenging (Jindal-Snape, Symonds, Hannah & Barlow, 2021). One reason for this is that students often come from different learning environments, bringing with them different levels of knowledge and different perceptions of mathematics. Moreover, students' prior mathematical knowledge often focuses on calculation, formulas, and memorization (Symonds & Galton, 2014).

Recent discussions about teaching and repetition in Swedish schools are strongly rooted in educational history and teaching culture (Bergsten, 2002; Östergren, Träff, Elofsson, Hesser, & Samuelsson, 2023). Historically, students often learned subject–specific content through counting and repetition of prior knowledge in isolation, without a connection to new knowledge (Karlsson & Kilborn, 2023a, 2025). In Swedish primary schools, students' learning of mathematics often focuses on rote memorization, for instance, of multiplication tables, facts, and formulas, instead of promoting conceptual learning (Karlsson & Kilborn, 2023b). At the same time, rote learning without an understanding of mathematical concepts is meaningless, as it cannot be carried over to a systematized context or extended into new knowledge. The same issue has been highlighted by several international researchers such as Katzoff, Zigdon, and Ashkenazi (2020), as well as Yuliandari and Anggraini (2021). This was also illustrated in a survey study of pre-service teachers' prior mathematical knowledge in relation to a repetition test at the start of their mathematics studies in teacher education. The study clearly showed that knowledge gained through rote memorization is not available and difficult to apply when solving text-based and problem-solving tasks (Karlsson, 2015). Learning with a focus on low cognitive domains, such as memorization and calculations, does not lead to mathematical generalization or an understanding of mathematics (Smith & Stein, 1998).

Moreover, primary school mathematical activities are mostly dominated by lower-level cognitive learning, rote knowledge, and numeracy (Carmichael, 2015). This can lead to problems when students repeat arithmetic operations or when algebra and functions are introduced, which may have consequences for their continued studies of mathematics. To resolve this, it is necessary to activate, systematize, recapitulate, and extend previous learning through optimal conceptual teaching. Hence, repetition based on students' prior knowledge, applied in a new, broader context, is a key factor in organizing students' knowledge, understanding, and relational thinking (Skemp, 1976). According to Brinkmann (2017), repetition is a didactical strategy in teaching that aims to actualize and extend the knowledge that students already have or had in the past.

It is not sufficient for students to work with mathematical tasks they have never understood. A study of students' learning of operations with rational numbers in grades 7–9 showed major deficiencies among students in grade 7 (Baldemir, Tutak, Nayiroglu, & Suzen, 2023). Moreover, studies such as Avcu (2024) illustrate that students face challenges in learning rational numbers. Many of these challenges were still present in grade 9, where most students had insufficient knowledge or lacked conceptual knowledge of fractions and related concepts. As a result, students had limited opportunities to transfer knowledge from primary to secondary school, which impeded the development of new conceptual knowledge of algebra and functions. In other words, continuous repetition, supported by teachers during the teaching process, is required for students to assimilate previous knowledge and establish a framework for learning new knowledge. Knowledge transfer, specifically from primary school to upper secondary school and from arithmetic to algebra, continues to be an important area of research (Matiti, 2022).

Before introducing new learning content, it is important to ensure that students have the prior knowledge necessary to assimilate new mathematical concepts or methods. For some students, it may be enough to recapitulate and systematize what they have already learned (Kania, Fitriani, & Bonyah, 2023). For other students, the gap between prior knowledge and new knowledge to be assimilated is sometimes extensive and requires a new approach to teaching that integrates two interwoven parts: a core based on prior knowledge and a component based on new knowledge. Helping students continuously gain access to relevant mathematical concepts and methods is crucial for teaching in a democratic classroom (Stemhagen & Henney, 2021). To minimize gaps in students' prior knowledge, it is recommended that teachers analyze and review which previously learned content is most important and provide balanced, conceptually grounded teaching. This type of teaching is essential to ensure equal learning opportunities for all students. According to Vygotsky, students' learning occurs through the integration of prior knowledge and new knowledge within a foundation of conceptual continuity, which is crucial for students' knowledge development (Vygotsky, 1978, 1987). Against this background, it is natural to interpret the meaning of repetition in relation to the meaning of the learning process, not just as a reconstruction of necessary, but not yet acquired, knowledge and skills.

The purpose of this study is to examine how students' repetition of arithmetic concepts is organized in teaching during the first year of upper secondary school in order to extend research focusing on the conditions for students' outcomes in algebra through repetition processes and on the conditions for students' knowledge transition from arithmetic to an algebra learning context. More specifically, the study seeks to address the following research questions:

RQ1: How do teachers organize the repetition process of arithmetic concepts before and within the teaching of algebraic concepts and functions?

RQ2: How do teachers assess students' prior and new knowledge, and what conceptual support do they draw from the relevant textbook when developing a method for repetition*?*

2. THEORETICAL FRAMEWORK

The theoretical conceptual design draws on Freudenthal's (1973, 1983) didactical phenomenology of mathematical structures and Bernstein's (1996) repetition-without-repetition learning theory. This design was used for the conceptualization of repetition in mathematical instruction.

Freudenthal (1983) highlighted the importance of understanding mathematical concepts, their conceptual meaning, and context. A fundamental didactical and phenomenological analysis of mathematical concepts and structures makes it possible to define mathematics as a science and understand its use and application, particularly in teaching. Understanding the phenomenology of "mathematical structures, knowledge of mathematics and its applications" through didactical phenomenology provides knowledge that can be applied in instruction and in monitoring students' cognitive development and growth. According to Freudenthal (1983), mathematics cannot be taught without ensuring continuity in students' learning and activating their prior experiences. At the same time, mathematical instruction must include realistic mathematical activities (Freudenthal, 1991; Gravemeijer, 1994; Tall, 2004) that provide students with the prerequisites and conditions to learn mathematics according to their own abilities and limitations.

A crucial aspect of Freudenthal's (1983) approach to teaching practice is the emphasis on the individual learning process and on students' learning needs in relation to the individual development and progression of knowledge. The significant impact algebra has on learning is the result of the conceptual expansion of arithmetic and the "intuitive" methods primarily used by younger students. Teaching elementary arithmetic as numbers and their operations is key to practicing algebra and using algebraic rules and methods (Freudenthal, 1973). At the same time, algebra mastery occurs on another level of knowledge and requires renewed practice of algebraic methods, as "traditional methods from elementary arithmetic" are not enough. According to Freudenthal (1973), arithmetic and algebra are fundamentally different because arithmetic is "intuitive and close to reality or at least it should be so" (p. 287). Algebra, however, is abstract and grounded in "formal-symbolic methods." To minimize the conceptual difference between arithmetic and algebraic methods, it is important to introduce and apply algebraic methods at an early stage, when students are first introduced to algebraic calculus and algebraic symbols. This kind of didactics can be used, for example, in students' learning of algebra by illustrating how algebraic methods build on the arithmetic they learned in previous grades. In other words, there is a need to create a bridge between students' prior arithmetic knowledge and new algebraic knowledge, while also incorporating conceptual repetition. This means that the learning of arithmetic in early grades can be organized to create conceptual continuity from whole numbers to rational numbers, and so on. Teaching and learning algebra, as well as the development of students' mathematical skills, is a multifaceted process. At the same time, systematic repetition is essential for providing conceptual continuity and supporting long-term teaching and learning (Tall, 2004). A teaching approach that incorporates principles of repetition can help students systematize and reproduce their knowledge of mathematics, understand the challenges they encounter, and activate effective learning strategies (Gray & Tall, 1994).

Bernstein's (1996) concept of "repetition-without-repetition" (RWR) emphasizes that the repetition process needs to be implemented in teaching continuously, "by every movement," through educational tasks consisting of new learning components (Palatnik, 2023; Shvarts & Abrahamson, 2023). According to Bernstein, repeated practice leads to new ways of thinking.

The conceptual framework for repetition is knowledge development. Bernstein adds that "absolute repetition" (one-by-one) cannot be part of a learning process that aims to develop knowledge successively. This theoretical framework is used in this study to define the conceptual meaning of repetition as a teaching phenomenon and is seen as a logical extension of Freudenthal's (1973) phenomenology.

3. RESEARCH DESIGN

The study was designed with the aim of examining teaching practices in relation to students' learning of algebra and functions, based on how teachers implement the repetition process and the conditions that promote the transfer of students' prior arithmetic knowledge into new knowledge about algebra and functions.

Participants included three upper secondary school mathematics teachers and their three classes, comprising a total of 94 students (34 students in class A, 34 in class B, and 26 in class C).

First, it was important to establish a constructive discussion with the teachers about the teaching process, the structure of the content, and planning for repetition to support students' learning through repetition, by linking new knowledge to prior arithmetic knowledge. This was arranged through discussions between researchers and teachers, based on the researchers' PowerPoint presentations and an analysis of tests intended as a basis for the repetition of arithmetic and algebraic concepts. The tasks were selected from the relevant textbook (Szabo et al., 2021).

Second, the collected data were used to create a holistic picture of the repetition process and teaching focusing on students' learning, as represented by the conceptual design for the study in the Figure 2 below, in Section 4.1, "Conceptual design for teaching and learning."

The researchers designed a qualitative methodology, including group discussions with the teachers and individual discussions about tests used as a basis for repetition. This led to reflective, interactive discussions between researchers and provided insight into teachers' development through a reflective process.

The key arithmetic content for repetition included whole numbers, rational numbers, and powers while the relevant teaching content for algebra included algebraic expressions and linear and rational equations. The main component for arithmetic repetition and algebra teaching was Freudenthal's (1973) didactical phenomenology of whole and rational numbers, as well as his perspective on the number concept, applied arithmetic, and the development of number concepts into algebraic methods. A theoretical conceptualization of inverse operations and algebraic concepts was also used (Karlsson & Kilborn, 2024), as well as the didactical structure for teaching arithmetic, algebra, and functions. The standard arithmetic repetition test, constructed and used by teachers, is illustrated in Figure 1.

Figure 1.
Standard arithmetic repetition test.

1) Solve: **1a**. 2 + (-4); **1b**. (-3) + (-7); **1c.** 4 – (-2); **1d.** (-6) – (-4); **1e.** (-7) + 5 – (-4); **1f.** (-6) + 3 – 4.
2) Write in exponential form: **2a.** 575 000 000; **2b.** 0.000 04; **2c.** 301.2.
3) Solve: **3a.** 24 – (-8 – 13); **3b.** 5 – 2·(-3).
4) Solve: **4a.** $\frac{1}{5}+\frac{3}{5}$; **4b**. $\frac{2}{3}+\frac{1}{6}$; **4c.** $\frac{2}{5}\cdot\frac{3}{4}$; **4d.** $12\cdot\frac{2}{3}$.
5) Solve: **5a.** $7^3 \cdot 7^2$; **5b.** $7^2 \cdot 7^{-5}$; **5c.** $7^2 + 7^1 + 7^0$; **5d.** $7^{-7}/7^3$.
6) What number is x? $3^7 \cdot 3^4 = 3^x$.
7) Solve: **7a**. 5^{-2}; **7b.** $3^2 \cdot 4^{-2}$.
8) Solve: **8a.** $4^{\frac{2}{3}} \cdot 4^{\frac{4}{3}}$; **8b.** $\sqrt{9} + 9^{\frac{1}{2}}$.
9) Solve in potencies form: **9a.** $\frac{(7^3)^2\cdot 7^4}{7^5}$; **9b.** $\frac{100}{0.001}$.
10) Solve: **10a.** $2.51 \cdot 10^3$; **10b.** $3 \cdot 10^{-2}$; **10c.** $4.23 \cdot 10^{-7}$.
11) Solve: $(25^{\frac{1}{2}})^3$.
12) Write in base 5: $25^8 \cdot 5^{11}$.
13) Simplify as far as possible: $2^{-3} + (\frac{3}{2})^3$.
14) Solve the equation: $2^n + 2^n + 2^n + 2^n = 2^{30}$.
15) Solve in exponential form: $27^{-2/3}$.

A mixed-methods methodological design was used in the data collection process. A *quantitative* analysis was used to examine the content of teacher–researcher discussions, researchers' participatory observations, reflective journals, pre-tests prior to repetition, arithmetic-algebraic-functions learning tasks, and analyses of learning tasks in the relevant textbook.

The joint discussions provided insights into teaching practice, teachers' experiences and perceptions of practical classroom work, as well as their knowledge of teaching from their own teacher–education programs. In the present study, the key components of the methodology were also the *qualitative* analysis of the collected data and the research questions, which aimed to identify significant empirical findings. A qualitative analysis includes teachers' written teaching plans concerning arithmetic repetition and algebra-function content, the analysis of students' answers to a repetition test in a focus group, and the analysis of teaching content such as teaching-learning tasks and the repetition content in the textbook in use. To further analyze students' performance on the repetition test, a focus group of 10 randomly selected students was constructed. Their answers to the respective tasks in the repetition test were examined both *quantitatively and qualitatively*.

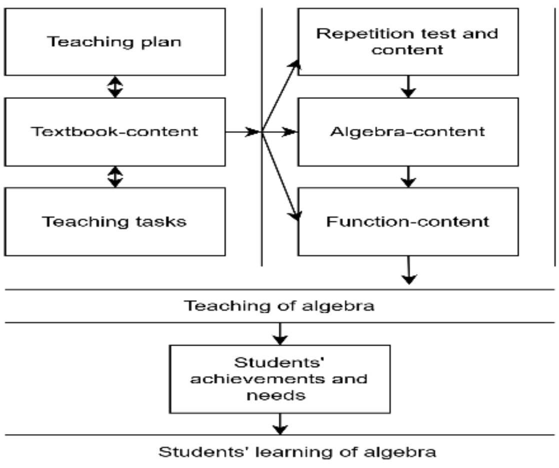
To optimize the process, the researchers actively participated in discussions on the planning of teaching and learning tasks, helping teachers systematize and reflect on the relevant content. Researchers also contributed PowerPoint presentations and information about current research, as well as its practical implementation in new teaching strategies, teaching methods, and content.

The methodology added conceptual meaning to the study, addressing subject-specific repetition, the presentation of new content, the profound conceptual learning of arithmetic-algebraic-function concepts, and the crucial connection between concepts in teaching and learning (Chick, Baker, Pham, & Cheng, 2006). The researchers also examined the practical application of repetition in the classroom environment in relation to the applicable design and methodology, as well as arithmetic-algebraic teaching and learning content, repetition tests and tasks, and their relationship to students' learning of new content (Timperley & Alton-Lee, 2008). The research methodology led to a conceptualization of the repetition process that can provide a spectrum of outcomes, including the expansion of students' prior arithmetic knowledge and the development of a theoretical framework that contributes to algebraic learning.

3.1. Conceptual Design for Teaching and Learning

The conceptual design developed by the researchers and used in the study is illustrated in Figure 2. The design forms the basis for the structure of the results section.

Figure 2.
Conceptual design of the study.



The conceptual design for the study, considering the repetition process and students' learning, is rooted in Bernstein's repetition-without-repetition (RWR) approach.

3.2. Ethical Considerations

Participants were informed about ethical principles, including trustworthiness and anonymity. Data from teacher discussions were treated confidentially, and conclusions regarding instructional planning were based on objective sources such as written lesson plans. Students' arithmetic test results were anonymized. At the start of the study, teachers were briefed on procedures for collecting, interpreting, and using qualitative and quantitative data for research purposes. The subject-specific content of the textbooks was discussed with teachers in relation to conditions for students' learning of algebra through repetition, framed within a clear research-theoretical context. Ethical standards followed the Swedish guidelines for Good Research Practice (Swedish Research Council, 2024), which outline principles for research integrity and ethics.

3.3. Analysis

The collected data were categorized in relation to the research questions and interpreted carefully. Teachers' reflections on their experiences and perceptions of practical classroom work were interpreted in relation to teaching current content, mostly based on a written teaching plan, how teachers chose teaching-learning tasks, including tasks in the textbook and tasks in the repetition test, as well as their analysis of students' results in the repetition test and subsequent planning of teaching. The focus of the analysis was the current teaching, including the written teaching plan and the content of the repetition test, teaching-learning tasks, and teachers' analysis of students' achievements, needs, and knowledge development.

4. FINDINGS

4.1. Finding 1. RQ1

How do teachers organize the repetition process of arithmetic concepts before and within the teaching of algebraic concepts and functions?

At the start of the study, researchers noted that teachers interpreted repetition in teaching as the number of weeks prior knowledge is reviewed. This review period begins with a common test used by all teachers in the same municipality, followed by repetition tasks from the textbook.

As mentioned earlier, teachers noted challenges in teaching subject-specific content based on students' outcomes on the test. When researchers asked about students' knowledge, the teachers' answers were consistently based on the content in the textbook. Over time, however, teachers were able to approach teaching from a broader teacher–student perspective.

Arithmetic repetition lasted for five weeks: one week for whole numbers, one week for fractions, one week for decimals, and two weeks for exponents. At the end of the fifth week, the students completed the standard arithmetic repetition test. The written teaching plan for repetition was taken from the first chapter of the relevant textbook and had a variety of goals regarding what students should review and why.

Repeated discussions with teachers revealed that their understanding of repetition was inconclusive and unbalanced, particularly concerning what students should be taught during the repetition phase and how it should support their acquisition of new knowledge. Some of the students had poor results on the repetition tests, which will be discussed later, but this was never noted, and no measures were taken to teach the content after the repetition test based on the students' results.

The teaching and teaching plan were tied to the structure and content of the textbook used. In discussions with the teachers, it became clear that they always follow the applicable teaching plan and the repetition test.

At the end of the study, researchers noted that the teachers' perceptions of repetition remained focused on the repetition tasks in the textbook rather than a plan for teaching or content goals. Teachers attributed the students' poor performance to textbook content and their previous learning experience. Teachers did not acknowledge their own role in students' learning or their inability to develop an effective learning process. Although five weeks were set aside to revisit and reinforce students' prior knowledge, instruction was dedicated to the repetition of tasks in the textbook with no relation to students' real needs. As a result, the students' misconceptions were often reinforced, rather than learning strategies to acquire and apply the necessary knowledge. The lack of clear goals and selection of inadequate repetition tasks, combined with the teachers' organization of teaching, had negative consequences for students' acquisition of new knowledge.

The discussions with teachers provided insights into their approach to teaching algebra and functions, revealing that this content was seen as an isolated task. Furthermore, teachers lacked effective strategies to apply, update, and integrate arithmetic repetition content into current teaching and learning. Despite assistance from the researchers, who suggested new teaching approaches to create better conditions for the development of new knowledge of algebra and functions, no changes were made immediately. The teachers' explanation was as follows:

[...] we had a number of relatively broad questions about teaching and learning algebra and [...], we have received guidance and help during the year (2024–2025) to sort and narrow down the different questions and structure possible areas of development going forward. It has been very inspiring and helpful for us to receive this research guidance and help, but above all it has given us new perspectives and tools to develop and think about new ways of teaching.

The teachers' plan for instruction continued to treat algebra and functions as separate entities, without a relational and logical relationship to arithmetic. A likely explanation for teachers' inability to change their approach is that their own education was not adapted to research in mathematics education or contemporary changes in teaching and schooling. An important conclusion is that teachers' actions must be evaluated in relation to students' outcomes and their opportunities to develop new knowledge in an environment that supports knowledge acquisition. The teachers' written reflection was that:

[...] we have had confirmation that many parts of our way of teaching today are in line with what research has shown is effective for students' learning. A concrete activity that has been particularly rewarding during the research has been when we have analyzed mathematical tasks for teaching and learning together with researchers and analyzed repetition test results and created/analyzed diagnoses to follow up on students' learning in a specific area (there has been a particular focus on algebra).

This means that teachers appreciate the development of a new teaching-learning approach that took place together with researchers. Another important point is that teachers need to understand and develop the teaching process through continuous reflections and actions based on how the students have mastered the content. This takes more time than the timeframe for this study, but teachers have good theoretically based presumptions from the current study.

The results show that teachers did not optimally and effectively adapt teaching content before and consider its effect on students' knowledge acquisition during repetition. Against this backdrop, the results highlight the importance of a balanced learning environment, grounded in comprehensive teacher education and a supportive teaching culture within schools.

4.2. Finding 2. RQ2

How do teachers assess students' prior and new knowledge, and what conceptual support do they draw from the relevant textbook when developing a method for repetition?

Results from the standardized test (Figure 1) given after repetition are presented in Table 1. A total of 63 students participated, and the test consisted of 34 problems.

Table 1.
Standard arithmetic repetition test and students' results
(T = task and CA = correct answer).

Whole number and operations								
T	1a	1b	1c	1d	1e	1f	3a	3b
CA	92%	87%	76%	68%	48%	48%	41%	33%

Fractions and operations				
T	4a	4b	4c	4d
CA	94%	83%	71%	55.5%

Potencies and operations								
T	2a	2b	2c	5a	5b	5c	5d	6
CA	70%	44%	41%	89%	68%	22%	55.5%	49%
T	7a	7b	8a	8b	9a	9b	10a	10b
CA	44%	17%	22%	48%	63%	16%	76%	84%
T	10c	11	12	13	14	15		
CA	44%	17%	16%	13%	13%	3%		

In order to interpret the quantitative results from Table 1, a random sample of 10 students' arithmetic test results, similar to a focus group, was selected. This is illustrated in Table 2.

Table 2.
Standard arithmetic repetition test and focus group results
(T = task and CA = correct answer).

Whole number and operations								
T	1a	1b	1c	1d	1e	1f	3a	3b
CA	70%	90%	90%	90%	50%	50%	40%	30%

Fractions and operations				
T	4a	4b	4c	4d
CA	80%	60%	40%	60%

Potencies and operations								
T	2a	2b	2c	5a	5b	5c	5d	6
CA	60%	50%	40%	80%	70%	40%	50%	70%
T	7a	7b	8a	8b	9a	9b	10a	10b
CA	30%	30%	10%	10%	40%	10%	80%	80%
T	10c	11	12	13	14	15		
CA	10%	-	-	-	-	-		

The main difficulties for the students in the focus group (correct answer frequency 50% or less in Table 2) were related to operations with negative numbers (tasks 1e, 1f, 3a and 3b), the multiplication of fractions (4c), converting decimals to exponents (tasks 2b and 2c), and operations with exponents (tasks 5c, 5d, 7a, 7b, 8a, 8b, 9a, 9b, and 10c). Tasks 8a, 8b, 9b, and 10c are related to operations with exponents using negative exponents.

The results illustrate that 15 out of 34 tasks in table 1 and 12 out of 34 tasks in Table 2 had correct answers from more than 50% of students.

A careful analysis of the repetition test could be used as a basis for identifying students' learning needs and difficulties in basic arithmetic. Students' current knowledge is an important component of teaching praxis and forms the basis for the effective implementation of arithmetic repetition, as well as the teaching of algebra and functions. However, the findings from the study show that these were not considered in the teachers' instruction.

Despite researchers presenting the results above in tables 1 and 2 and offering concrete recommendations for developing teaching based on students demonstrated arithmetic knowledge, no changes were implemented.

The teaching of algebra and functions continued without consideration of students' prior knowledge. It was apparent that the teachers needed significantly more time to develop and advance in how to analyze and use the textbook's tasks in the teaching process related to the learning of algebra.

A comparative analysis of the results in tables 1 and 2 yielded the same conclusions. Students struggled with tasks that are important for their continued learning of algebra and functions, namely operations with negative numbers, fractions, and exponents with a negative exponent.

This finding showed that the construction of knowledge has a logical conceptual connection, which was clearly illustrated in the results of the repetition test. Students' inability to understand negative numbers led to negative consequences for further knowledge acquisition, in this case, exponents with negative exponents.

The findings from tables 1 and 2 indicate that teachers need to consider how gaps in students' prior knowledge can be addressed in teaching. The results also show the importance of students' ability to master algebra and functions in a progressive way. It is important that teachers are aware of fact that students' learning is a dynamic process that depends on the instruction they receive.

A total of 63 of 94 students participated in the repetition test, and most demonstrated engagement while completing the tasks. Students' difficulties with the tasks during the five-week repetition period likely relate to previous experiences with learning mathematics.

Textbook and repetition tasks

The textbook has a special 43-page chapter for the repetition of arithmetic tasks, aimed at systematically repeating prior knowledge before algebra is introduced. The subheadings are: Whole numbers, Fractions, Decimals, and Exponents.

The number of tasks intended for this repetition is shown in Figure 3 below. The subheadings (levels 1, 2, and 3) in the textbook show the complexity of the tasks.

The quantitative mapping of the tasks for the repetition of arithmetic shows that the number of tasks on whole numbers, fractions, and decimals is small compared to the number of tasks on exponents.

The same tendency was found in the teachers' test given after the repetition. More emphasis should be given to level 1 arithmetic tasks rather than the equal distribution of tasks between levels and the progressive difficulty of the tasks (see Figure 3).

Figure 3.
Number of repetition tasks in the textbook.

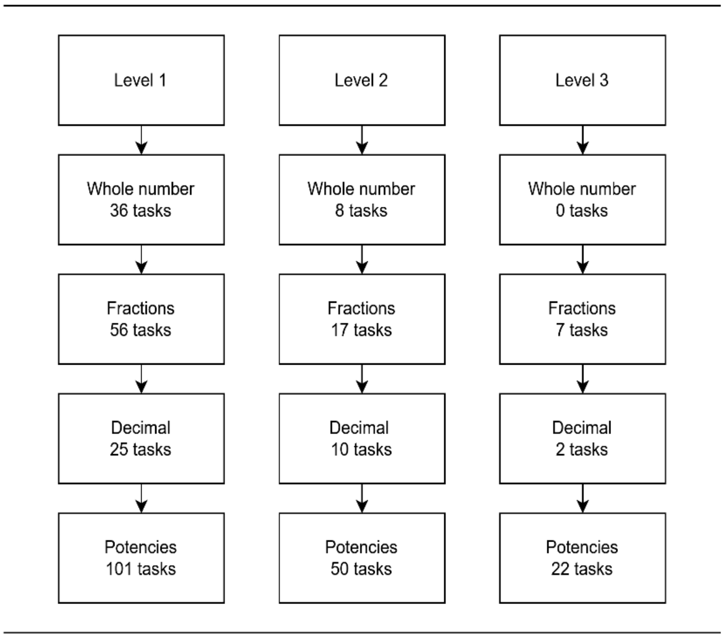


Figure 3 clearly illustrates that the number of tasks in each area and level are varied, though it is not possible to determine the reason for this distribution.

Textbook and repetition of mathematical content

The arithmetic tasks in the textbook have no clear or logical conceptual progression from level 1 to level 3 in relation to the focus and structure of the concepts.

For example, level 1 (p. 14) includes 10 tasks about reusing and extending fractions.

Level 2 contains one task on the lowest common denominator and two tasks on comparing the size of fractions.

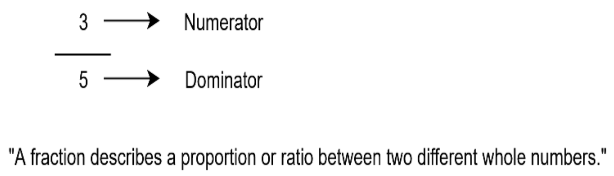
Level 3 includes just one task on the lowest common denominator. There is no logic in this. For example, a comparison of fractions is used to identify the lowest common denominator and should appear at level 1.

This is a poor conceptual transfer from one level to another, and there are similar tasks on level 1 in the example above.

The next question is how definitions of mathematical concepts are introduced in the textbook, for example, the concept of a rational number (fraction). In the textbook, fractions are introduced through a real-world situation, for example: "Three people in a family of five have decided to go on vacation.

This can be expressed with a fraction by writing that: '3/5 of the family has decided.'" This is then illustrated as shown in Figure 4.

Figure 4.
An interpretation of the definition of a rational number or fraction in the textbook.



The definition (an interpretation from the textbook in use) provided in Figure 4 is not correct. A fraction is described only through two selected properties and by a *proportion* or a *relationship* between two whole numbers. The same proportion can be expressed in different ways. In fact, there is no definition at all. In this case, the proportion 3/5 is described only by using a fraction. How should a fraction like $\frac{3}{5}$ be defined? Moreover, the concluding text is reversed: it is a proportion or a ratio between two whole numbers that can be described using a rational number, not the other way around. Another question is how a fraction like $\frac{6}{5}$ could be described with a real-world situation based on the "definition" in the textbook.

As an example, a correct definition of fraction is $\frac{a}{b} = a \cdot \frac{1}{b}$, where $b \neq 0$. To understand this, it is important to know that $\frac{1}{b}$ is the inverse of *b*. This is important in the next step in order to ensure a conceptual understanding of the formulas for multiplication or division of two fractions.

A qualitative analysis of the repetition tasks shows that most of the instructions for repetition describe *how* to accomplish the tasks, not *why* or *when* they should be accomplished.

5. CONCLUSION/DISCUSSION

This study examined how students' repetition of arithmetic concepts was organized in teaching during the first year of upper secondary school. Many of the findings highlight the importance of research-based teaching responses to repetition. The understanding of how teachers' repetition practices influence students' ability to repeat and systematize prior knowledge, as well as to apply prior knowledge in new learning, is central. The study clearly shows that the teaching process for repeating prior knowledge is not sufficiently developed and does not fully support students' knowledge acquisition or continuity in learning (Bernstein, 1996; Palatnik, 2023; Shvarts & Abrahamson, 2023).

The findings also support the research of Gray and Tall (1994), who highlighted the importance of learning effective strategies as a result of a solid teaching plan and content. Furthermore, Freudenthal (1973, 1983) supports the notion that students' knowledge of arithmetic is a key for learning algebra and the corresponding abstract aspects of mathematics.

The study also has practical implications for teachers, highlighting areas of teaching practice that must be improved and developed, such as addressing students' prior knowledge through a continuous learning process that integrates both prior knowledge and new knowledge (Bernstein, 1996; Tall, 2004).

In terms of content repetition, the study clearly shows that shortcomings in prior knowledge lead to problems in students' continued learning. (Freudenthal, 1991; Timperley & Alton-Lee, 2008). In this context, it is not beneficial for students to repeat algebraic and function concepts if teachers lack progressive strategies for improving students' knowledge through repetition.

The study shows that the balance between teachers' mathematical content knowledge, the choice of repetition content, learning-based teaching, and the careful, analytical use of textbook tasks in teaching creates a crucial "mind-map" for the repetition process, providing a foundation for achievement and knowledge development (Chick, Baker, Pham, & Cheng, 2006; Karlsson & Kilborn, 2024).

Students' results from the repetition test, combined with a number of ambiguities in the textbook and vague concepts and definitions, provide insight into the current conditions for learning. There is a significant divide between teaching content, the content in the textbook, and students' conceptual learning. This clearly shows the importance of teachers' mathematical content knowledge, their ability to plan instruction, analyze students' learning and needs, and, not least, carefully analyze the content in the textbook. The results show that there is an imbalance between the necessary conceptual base students require as prior knowledge and the actual teaching and support from the textbook. This can have a number of negative effects on students' learning of algebra and functions.

One significant advantage of this study is its theoretical and methodological design, which incorporates psychological variables and triangulation in terms of a learning-oriented approach, repetition goals, and the integration of these learning goals into the teaching and repetition of arithmetic content, including its interplay with algebraic and function content. At the same time, it requires certain conditions and the optimal choice of methodology.

This empirical study should be continued, both in an analytical way and by developing a more optimal basis and methodology, including optimal qualitative methods for researching teaching and learning. In addition, teachers' active role and their ability to reflect on and adopt new teaching approaches are essential in this type of research.

In their reflections and discussions, teachers clearly indicated that they face significant challenges related to pedagogical knowledge, such as mathematical content knowledge for teaching. This concerns, for example, how to break down subject-specific content for instruction and how to organize teaching according to students' knowledge development. This has a significant impact on students' overall learning beyond the repetition process. This study highlights the importance of developing an alternative, learning-oriented approach to teaching. The findings from this study are intended to support the preparation of prospective teachers.

At the request of the teachers, this study will continue from the fall of 2025 through the fall of 2026, with a special focus on the design and construction of algebraic tasks based on students' learning.

6. LIMITATIONS

The first limitation of the study is that it was conducted in just one upper secondary school, which provides limited insight into other teaching environments. For this reason, the findings cannot be generalized to other school environments and teaching traditions related to repetition. The second limitation of the study is that discussion questions were interpreted by researchers without classroom observations.

Adding an observation component in subsequent studies could provide broader guidance for understanding repetition practice in schools.

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