Chapter # 9

UNDERSTANDING OF PRE-SERVICE TEACHERS' MATHEMATICAL KNOWLEDGE AND PERCEPTIONS OF TEACHING

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ABSTRACT
The objective of the present study is to analyze pre-service teachers' mathematics knowledge base as a relationship between their substantive subject mathematics knowledge and syntactic subject mathematics knowledge. The theoretical approach refers to Shulman's pedagogical content knowledge and Ball, Thames and Phelps' subject matter knowledge for the teaching concept. The participating pre-service teachers (PSTs) were divided into two groups: 20 from a program course for K–3 teachers and 25 from a program course for grade 4–6 teachers. The crucial mathematical content for the study was algebraic patterns, rational numbers (fractions), proportionality and elements of combinatorics. The findings show the importance of gaining a better insight into and understanding of student teachers' learning process and challenges related to the mathematics knowledge base (e.g., its substantive and syntactic components). The benefits of the empirical findings are that they led to new knowledge and a deeper understanding of PSTs' learning of mathematical concepts through two learning processes (SUB and SYN), as well as the relationship and shift between these processes.

Keywords: preparation of pre-service teachers, mathematics knowledge base, substantive subject matter knowledge, syntactic subject matter knowledge, mathematics for teaching.

1. INTRODUCTION

Mathematical preparation of PSTs plays a crucial role in developing their ability to teach mathematics and has a strong impact on their students' learning of mathematics. The quality of teacher education is an important topic of research in both a national and an international context. There is a need to expand our knowledge of PSTs' learning of mathematics, how to activate and systematize their prior subject knowledge, how to foster the transition from prior knowledge into knowledge for teaching mathematics, and how the interplay between theoretical knowledge and knowledge of practical teaching should be organized (Wilson, Floden, & Ferrini-Mundy, 2001). Wilson et al. (2001) also report that PSTs' knowledge of teaching content and practical teaching is often separated from, or poorly coordinated with, the theoretical part of their education. Student teachers' perceptions of mathematics teaching are often limited to mechanical learning (counting). One of the major challenges is creating continuity in PSTs' learning of mathematics, in which theoretical knowledge is developed in close connection with knowledge of mathematics teaching in school. A key prerequisite for achieving this is to find practice schools where teachers share the teacher educators' views of knowledge and learning (p. ii). Recent studies (Radovic, Black, Williams, & Salas, 2018; Zazkis. Leikin, & Chavoshi Jolfaee, 2011) confirm that there is still a need for a more comprehensive understanding of the kind of mathematics knowledge PSTs need to develop in order to enable high-quality teaching after they have graduated.

There is currently limited research on effective strategies to teach PSTs' mathematics and mathematics didactics in a way that can provide them with the necessary teaching skills through practical experience in schools, while also providing them with a deeper understanding of the crucial role of theoretical knowledge (Olawale, 2024). The combination of theory and analytical observation of teaching practice provides student teachers with the experience they need to access and understand mathematics, both for their own benefit and for the benefit of their future students. Access to theory and practice means that student teachers will have continuity in their mathematical development and in the planning of teaching content with a focus on students' learning. This means that PSTs and their future students will successively learn the internal structures of mathematical concepts through conceptual and procedural meaning which, in turn, must be generalized with the aim of understanding mathematics as an abstract subject. In order to implement such a process and ensure learning in the classroom, student teachers must be able to process and produce knowledge in relation to their own learning and their students' learning. In other words, they must be able to take a second-order perspective on their students' learning, which is intimately connected to a first-order perspective on their own learning of mathematics (Leatham, 2006). This requires self-awareness and an understanding of how mathematical knowledge is perceived and developed, and of the misconceptions that can arise in their own learning and their students' learning. Much like incorrect generalizations during their early school years, students' misconceptions of mathematical concepts and methods can have a significant impact when they reach lower and upper secondary school, as well as teacher education (Kajander, 2005; van Dooren, De Bock, & Verschaffel, 2010). In order to promote students' learning of mathematics and identify any misconceptions, PSTs must have solid knowledge of mathematics that is rooted in practical mathematics teaching.

2. BACKGROUND

The importance of mathematical knowledge and the conceptual context in which PSTs and their future students are taught has been described as *subject mathematics knowledge* for teaching by researchers such as Shulman (1986) and Ma (1999). Empirical studies of teachers' mathematical knowledge suggest that PSTs need to make a transition from a student's perspective to a teacher's perspective in order to engage in self-reflection on their own learning process of mathematical content, as well as understand how students learn the same content based on their different abilities, knowledge and, conditions for developing conceptual knowledge. In general, this research is relevant for PSTs' awareness of how they learn mathematics and mathematical content through educational activities in mathematics courses, as well as their observation and evaluation of mathematics teaching in schools, and their awareness and understanding of content when planning teaching activities. Research has also focused on the interconnection between PSTs' mathematical knowledge and reflections, and the analysis of teaching situations, which they may implement in collaboration with other teachers, or independently on the basis of their own experiences (Dewey, 1998). Psychological research has also looked at teachers' subject-related, content-specific professional development through critical reflection and self-reflection (Mezirow & Taylor, 2009; Hardy, Decristan, & Klieme, 2019). The present study focuses on PSTs' knowledge of teaching in combination with their reflections on practice and the analysis of the teaching experiences of their peers, as this constitutes a particularly important part of their own development as they form a mathematical identity. Understanding the continuous connection between the development of PSTs' knowledge of mathematics and teaching practices can provide a better understanding of the subject of mathematics, which will have a positive impact on individual competence and mathematics performance (Boaler, 2002). Moreover, PSTs who engage in positive teaching practices have been shown to achieve both academic and professional growth (Zhao & Zhang, 2017).

The opportunity to analyze the outcomes of conceptual teaching content in the practical teaching of mathematics enables PSTs to identify the relationship between practice and theoretical knowledge, as well as to construct conceptual knowledge. In particular, the ability of PSTs to evaluate the outcome of their practice may contribute to theoretical knowledge, or, as Shulman (1987) put it, that "the development of scientific knowledge for teaching as a diverse disciplinary knowledge and its interrelations as a framework for a knowledge base for teaching". In PSTs' mathematics education, the study of mathematics knowledge bases for teaching plays an important role, specifically in understanding the relationship between theoretical knowledge and teaching practice, as well as PSTs' learning of mathematics for teaching. This mathematics knowledge is key to the teaching of mathematics and for current research discussions about mathematics knowledge base and its impact on PSTs' learning to teach mathematics (Darling-Hammond & Bransford, 2005; Hordern & Tatto, 2018). The importance of mathematical concepts and their conceptual meaning related to PSTs' knowledge of mathematics has been highlighted by several researchers (Hine, 2015; Askew, 2020; Aguilar & Telese, 2018; Shulman, 2019; Weingarden, 2024) and in the context of STEM education (Alrwaished, 2024)

Leikin and Zazkis (2010) have claimed that the learning of mathematical concepts is a result of a subject mathematics content knowledge and an epistemological knowledge component. Empirical evidence of the connection between these components has led to a learning discourse on PSTs' mathematics education in which subject matter knowledge and content knowledge can serve as a logical starting point for understanding the planning of teaching activities, conceptual structures and teaching content (Anderson & Clark, 2012). As the formation of these concepts is considered crucial for teaching mathematics, researchers have indicated there is a positive relationship between subject mathematics knowledge and the quality of teaching (Ball, Lubienski, & Mewborn, 2001; Karlsson & Kilborn, 2021). This provides prospective teachers with an understanding of what to teach and how to teach it, while creating a context that supports student learning, continuity, and conceptual progression. PSTs' conceptual learning of mathematical concepts and concepts for teaching mathematics, as well as the transfer between them, is created on the basis of PSTs' knowledge of their future profession as teachers. This also includes conceptual learning specifically about subject mathematics in terms of the meaning of mathematical concepts, and the connection between mathematical concepts and mathematical teaching and learning (Jakobsen, Thames, & Ribeiro, 2013; Norton, 2019). This is an important part of PSTs' preparation, so that they can teach mathematics in a way that develops the knowledge their students need to understand mathematics and its link to science. This study examines and has an important focus on PSTs' learning of subject mathematics that deals with conceptual components and meaning regarding Shulman's theory of substantive and syntactic the structure of subject mathematics knowledge.

This study is a follow-up of previous research about PSTs' preparation for teaching mathematics (Karlsson & Kilborn, 2023a; 2024a; 2024b). The purpose of the study is to analyze PSTs' mathematics knowledge base and the meaning of conceptual development in their learning process related to substantive mathematics knowledge and syntactic mathematics knowledge, especially the interplay and shift between them. The study also addresses the interplay between the activities of practice-based PSTs' training practice and their impact on the development of substantive and syntactic knowledge. In general, the study investigates PSTs' learning of profession-specific knowledge for teaching mathematics in order to achieve an in-depth understanding of the development of professional skills for teaching mathematics. The study includes an entire section on PSTs' conceptual learning of mathematics for teaching in advance.

3. THEORETICAL CONCEPTUAL FRAMEWORK

The theoretical conceptual framework refers to the conceptualization of teachers' content knowledge for teaching, as defined by Shulman (1986, 1987). Ball, Thames, and Phelps (2008) have described pedagogical content knowledge and subject mathematics knowledge as a theoretical issue. In terms of skills and competencies for teaching, PSTs' content knowledge leads to professional prosperity and expertise through theoretical study and practical experience. To underline the significance of teachers' mathematical knowledge and the quality of (future) teaching, as well as its relationship to pedagogical content knowledge, subject mathematics knowledge and subject-specific content knowledge have been defined as categories for the development of a knowledge base (Shulman, 1986), in both theoretical terms and in the context of teaching practice. A mathematics knowledge base is intended to differentiate between knowledge of mathematics as a subject discipline (procedural meaning) and knowledge of mathematics and its application in teaching and learning (what is created through conceptual meaning) (Goulding, Rowland, & Barber, 2002). A knowledge base comprises two primary parts: definition and significance, and two kinds of knowledge, substantive (SUB) and syntactic (SYN). In theory, comparing these parts provides a conceptual overview of how mathematics should be understood and how it should be taught (Askew, 2020). SUB and SYN knowledge are important components of PSTs' mathematics subject matter knowledge in conjunction with their pedagogical content knowledge (Shulman, 1987; Zaslavsky, 2008; Venkat & Askew, 2018). Researchers argue that teaching mathematics is no easy task, and that preparing PSTs to teach mathematics is a complex process that requires careful thought and preparation. The development of PSTs' professional knowledge of teaching mathematics is tied to their knowledge of fundamental mathematics and its application via explorative, classroom-based experiences. In terms of SUB and SYN contexts, acquiring knowledge of mathematics means that PSTs learn the general principles for the construction of subject mathematics, that is, "knowing that" (SUB knowledge) and how they can organize teaching mathematics content (SYN knowledge) according to Turner-Bisset (1999). In a substantive sense, mathematics focuses on mathematical facts (e.g., mathematical structures, concepts and methods) and their relationship to the subject of mathematics, as well as essential properties and the relationships between facts, mathematical structures and concepts. Insofar as it requires epistemic knowledge, syntactic knowledge is about how mathematical structures and concepts originate from mathematical ideas. Syntactic knowledge has been developed throughout history and in different cultural contexts. SYN knowledge provides the foundation for knowledge about what and how to teach mathematics and for the development of teaching experience.

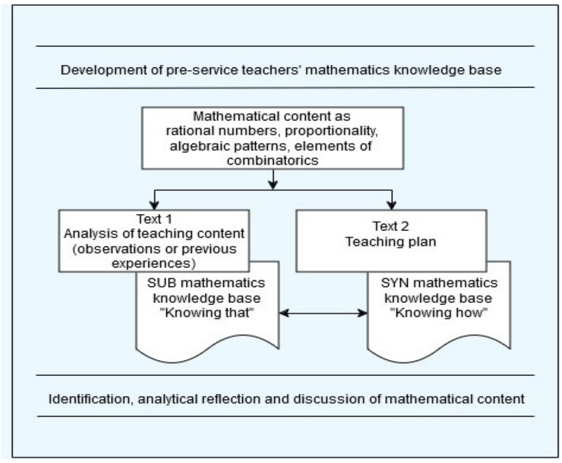
This study focuses on the structure of PSTs' substantive (SUB) and syntactic (SYN) knowledge, as this is fundamental to questions concerning what PSTs need to learn in order to effectively teach mathematics. The mathematical content that is crucial for this study includes algebraic patterns, rational numbers (fractions), proportionality, and elements of combinatorics. This is currently taught in mathematics courses.

4. METHODOLOGICAL DESIGN

4.1. Context and Participants

This study looks at substantive and syntactic knowledge in relation to primary school mathematics education in a course for pre-service K–3 teachers (Advanced Mathematics Development) and a course for PSTs of grades 4–6 (Mathematics 4). Both courses are bachelor' level programs. The study focuses on the PSTs' individual work, such as written analyses of observations and their design of a teaching plan. To track the development of the mathematics knowledge base of PSTs, the following conceptual chart has been used (Figure 1).

Figure 1.
PSTs' mathematics knowledge base.



Both courses were conducted in semester 6 of two primary school teacher education programs in the 2023 academic year. In both programs, most of the participants were female. Both courses lasted for five weeks, and the participants had no other courses during this period.

Pre-service K–3 teachers had four days of teaching practice, comprising observations and the analysis of teaching content. After the period of practicum, the PSTs were given two weeks to produce a written analysis of their observations. At the end of the course, they presented their detailed plans for teaching mathematics–specific content. The mathematical content the teachers could choose comprised algebraic patterns, rational numbers, proportionality, and basic combinatorics. For Text 2, the PSTs were asked to develop a design and mathematical content for a teaching plan. The mathematical content for Text 1 and Text 2 did not need to be identical.

The course for the grades 4–6 program did not include a special practicum component (period of practicum in school). Instead, the student teachers used their experiences from observations during an earlier period of practicum in school (part of a school-based training course). At the end of the course, the PSTs were asked to produce two written analyses (Text 1 and Text 2) (see Figure 1). The 4–6 student teachers could choose from the same crucial mathematical content as in the K–3 course. The PSTs from the K–3 and 4–6 programs differed in terms of their initial learning conditions, whether they had direct access to practicum work, and the amount of time allocated to complete Text 1. Special attention was paid to the fundamental differences and similarities between these learning conditions. The key component of the study was the analysis of the PSTs' development of substantive and syntactic knowledge, as well as the relationship between them (see Figure 1).

4.2. Data Collection and Analysis

The data include anonymized copies of the Text 1 and Text 2 assignments for individual student teachers: 20 PSTs' assignments from the K–3 program and 25 assignments from the grades 4–6 program (Corbin & Strauss, 2008). These data were analyzed using a conceptual knowledge base with both substantive and syntactic components in order to examine how the PSTs described knowledge in the substantive sense of "knowing that" (i.e., mathematical facts and concepts in Text 1) and in the syntactical sense of "knowing how," (i.e., the systematic analysis and evaluation of concepts in terms of teaching, argumentation, and logical ability in Text 2). The first component addressed the PSTs' analysis of teaching content, and the second component addressed their design of a teaching plan using a subject-specific mathematics content, see Figure 1.

4.3. Methods for Analyzing the Content in Text 1 and Text 2

It is important to note that the evaluation and analysis of written content were related to the PSTs' knowledge base. In Text 1, this was related to SUB structure and in Text 2, to SYN structure. The qualitative analysis of Texts 1 and 2 uses three criteria related to Marton and Booth (1997). For Text 1 (Analysis of teaching content), the following qualitative criteria were used: (1) *identification* of mathematical content; (2) *analytical reflection* on structure and the meaning of content; and (3) *analytical discussion* about how the observed teaching content could be developed. In addition to these three criteria, four new categories have been used: (1) understanding of instructions; (2) use of relevant theoretical discourse from course books (literature) and lectures in order to identify, analytically reflect, and discuss the observed teaching content; (2) correct mathematical language; (3) correct mathematical terminology; and (4) logical sense and clear structure in the written text. For the analysis of Text 2 (Design of a teaching plan), the following criteria were used: (1) *choice of* mathematical content concerning course content; (2) *analytical reflection* on the teaching plan, including two mathematical activities within the chosen mathematical content; and (3) *analytical discussion* about the connections between the teaching plan and the learning outcomes (more specifically, the concept and essential properties of the concept). For Text 2 course books, the school curriculum and textbooks were used for the respective grades. In such the same the above categories (1–4) were used.

4.4. Mathematical Content

The crucial course contents for Text 1 and Text 2 are rational numbers, algebraic patterns, proportionality, and elements of combinatorics. In this context, mathematical concepts are considered in relation to the theoretical discourse about "how to teach" and by "breaking down" the concepts in the teaching. The concept of rational numbers is related to sub-constructs (essential properties of fractions) as part-whole, part of a number, fractions as a number, fractions as a ratio, and the connection between the essential properties of a fraction with ratio, proportion, and proportionality, as well as operations with rational numbers. Algebraic patterns are related to the multiplication table and its structure and problem-solving tasks (Karlsson & Kilborn, 2023b). Proportionality is related to the proportionality constant, proportional relationships, proportionality, and problem-solving tasks. The elements of combinatorics are related to combinations and permutations and their representations via diagrams. The content was carefully presented in the course book (Karlsson & Kilborn, 2015) and during the lectures.

5. RESEARCH FINDINGS

5.1. Substantive Knowledge and Text 1

Overall, clear elements of a substantive component were identified in the assignments (Text 1) for 12 pre-service K–3 teachers and 10 pre-service grade 4–6 teachers. Among these, it was apparent that during the analytical processes, it was difficult for the PSTs in both groups to distinguish between conceptual facts and teaching and facts related to the subject of mathematics. Also, the PSTs had difficulty organizing a sequential description and analyzing concepts and struggled in the identification, reflection and discussion (SUB component) of mathematical concepts. Even though the observed teaching included elements of combinatorics or algebraic patterns, the written analysis tended to focus on limited, fact-based conceptual issues, primarily on natural numbers and counting with natural numbers. These results, which are described above, were found in the assignments (Text 1) for both the K–3 and grade 4–6 PSTs. It was apparent that it was difficult for PSTs to convey their own knowledge of subject mathematics (SUB component) to the observed teaching practice. Instead, they focused on the content they knew best, such as natural numbers and other easily understood content. They had different time frames for producing Text 1, but the differences had no discernable effect on the quality of their texts. The K–3 PSTs had two weeks to produce Text 1 (based on direct observations), and the grade 4–6 PSTs had five weeks to work on Text 1. Both groups faced the same challenges in demonstrating sufficient (substantive) knowledge. Most PSTs did not use the content from the relevant course books or lectures. It appears that they tried to address the teaching content but with no theoretical basis. This indicates that PSTs perceive teaching practice as a separate event with no connection to the content structure. It was interesting to note that the participants believed that their analysis/understanding of mathematics teaching in primary school did not require theoretical knowledge. They described the teaching of mathematical content from their own experiences and understanding of what teaching should look like and it was difficult for them to shift from practice to theory and vice versa. The results of the study show that the PSTs in both groups had poor content knowledge related to the SUB component, which was exacerbated by the participants' negative perceptions of theory and literature.

5.2. Syntactic Knowledge and Text 2

The analysis of Text 2, which comprised PSTs' detailed teaching plans, indicated a positive change in the achievement of SUB knowledge in both groups. 18 out of 20 K–3 and 16 out of 20 grade 4–6 PSTs were able to provide a theoretical account of a SUB component and showed a tendency to try to overlap SUB and SYN components in their texts. During the course, most of the participants learned conceptually toward transformative SUB-based thinking and were able to incorporate this into their teaching plan. However, it was still challenging for them to systematically assimilate the content in a teaching situation. It was difficult for them to change perspective, that is, to change the perspective of their analysis from a student's perspective to teacher's perspective and take on the role of a teacher. This led to them making a statement about mathematical facts, rather than the construction of a teaching plan. It was clear that SUB knowledge serves to enhance the qualitative development of SYN knowledge. These findings indicate that subject-specific content knowledge is key to the ability of PSTs to construct syntactic knowledge.

6. LIMITATIONS AND FUTURE RESEARCH DIRECTIONS

While this study reveals several key findings, some limitations should be taken into account. Firstly, because of the theoretical design of the study and the use of qualitative analysis, no general conclusions can be drawn. Secondly, as there only a limited number of participants, any further studies should use more participants. Thirdly, because the study did not examine the participants attitudes, behaviors and relationship to the subject of mathematics, or their participants' previous studies during their teacher education program, this should be taken into account in future studies as it may have a strong impact on the SUB and SYN knowledge of PSTs. Finally, the present study focuses on a mathematics knowledge base and a hidden process behind participants' learning of mathematics for teaching in the framework of the current study. Thus, the findings of the present study are not generalizable and are tied to the specific context of the study.

This research has highlighted the importance of the relationship between prospective teachers' preparation, mathematics knowledge base, and ability to learn and teach mathematics.

Further research is needed to clarify the optimal relationship between the development of theoretical knowledge of mathematics for teaching and the students' period of practicum, with a focus on the advantages and disadvantages. Research should also follow up on the quality of PSTs' mathematical preparation and promote innovative mathematical instruction for future teacher education.

7. CONCLUSION

The purpose of this study was to analyze PSTs' knowledge base and its development in relation to substantive subject matter knowledge and syntactic subject matter knowledge. Some interesting findings emerged from the study. Firstly, mathematical content such as algebraic patterns, rational numbers, and elements of combinatorics requires a certain level of conceptual knowledge in order to be assimilated by PSTs. Their performance was poor when it came to algebraic patterns and rational numbers. In their analysis, they tended to rely on the mathematical content they could recall from lectures and seminars, rather than using it combinatorically. In short, PSTs were able to replicate, but not create something of their own. At the same time, combinatorics was the best-described learning field for PSTs. Our hypothesis is that they learned combinatorics through experimentation (visualization), whereas algebraic patterns and rational numbers are more abstract forms of mathematics. This finding also indicates that PSTs' knowledge and early learning of mathematics still has a strong impact on their learning of both components, but it has a greater impact on the SYN component. The study highlights the importance of paying more attention to how PSTs learn mathematics for teaching. This is consistent with previous studies (Shulman, 1987; Turner-Bisset, 1999; Zaslavsky, 2008; Venkat & Askew, 2018), where researchers have emphasized that the process of preparing PSTs to teach mathematics is complex and requires proper forethought and preparation.

In summary, the results of this study reveal that student teachers face challenges in learning and applying mathematical concepts, as well as in their development of SUB knowledge. This means that in education programs for mathematics teachers, more attention should be given to developing PSTs' knowledge base in the form of SUB knowledge and the transformation from a SUB to a SYN mindset, as well as the impact of this transformation on student teachers' learning of mathematics for teaching (Goulding et al., 2002). The present study provides a broader understanding of the process by which student teachers learn mathematics for teaching. The results clearly show that the challenges related to the substantive and syntactic components of teachers' knowledge base need further investigation. The study also illustrates that the learning ability of PSTs is highly dependent on theoretical knowledge and that the school-based period of practicum was not sufficient for developing such a knowledge base.

The benefits of the empirical findings are that they led to new knowledge and a deeper understanding of PSTs' learning of mathematical concepts through two learning processes (SUB and SYN), as well as the relationship and shift between these processes. Thus, this research is very timely and raises some important points for future research as well as a broad range of theoretical and practical knowledge for teacher education.

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