Chapter # 5

TOWARDS A GENERALIZATION: WHAT STUDENTS LEARN ABOUT MULTIPLICATION

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ABSTRACT
This chapter examines and analyzes students' learning about aspects of the concept of multiplication with a focus on perceptions and representations, and how they apply this to handling multiplicative situations and patterns in the multiplication tables. The analysis has been performed in the context of the generalization process related to teaching activities, with a focus on students' perception of multiplication. The theoretical approach is based on Davydov's (1990) view of theoretical generalization as a perception-conception-elementary concept (PCE model). The current mathematical content was classified according to: (1) multiplicative structures (Vergnaud, 1983); and (2) basic laws of algebra (van der Waerden, 1971). The relationship between students' learning and the teaching process was studied in order to identify students' learning in action. The study comprises two teachers and 40 students in two classes in grade 3 and was followed up two years later in grade 5 with one teacher and 25 students. The findings of this study can provide knowledge about students' learning about multiplication using structures and multiplication tables in a conceptual context.

Keywords: multiplication, multiplicative structures, basic laws of algebra, teaching and learning multiplication.

1. INTRODUCTION

The development of instructions for promoting mathematical strategies, such as supporting teachers in understanding mathematical concepts, and when and how students are ready to learn such concepts, are essential prerequisites for students' learning (Ball, Thames, & Phelps, 2008). The guiding principle of teaching plays a key role in helping students to generalize essential-intuitive-primitive models of multiplication into general and more abstract models. Teaching activities guide students through the various stages of transforming a problem situation and identifying the crucial relationships within it. This constitutes the concept of multiplication as a basic level of multiplicative thinking (Kaput, 1985).

Previous studies have noted that students' early encounters with mathematical structures and patterns supports their mathematical development in subsequent years (Mulligan & Mitchelmore, 2009). A range of different conceptual dimensions of multiplication will also increase the opportunity for students to choose suitable strategies when dealing with multiplication situations (Thompson, 2017). The development of students' strategies and conceptual knowledge of multiplication are two coherent processes. The students' learning is also dependent on the teaching process and the content of teaching the concept of multiplication (Chin, Jiew, & Taliban, 2019). Another theoretical-empirical view is that a key element of students' learning of multiplication is the multiplicative structures and basic laws (multiplication) of algebra and how to apply them in different contexts. This can provide a conceptual basis for the students' learning of multiplication and algebra and help them to continuously process mathematical thinking (Karlsson & Kilborn, 2018).

Multiplicative thinking is one of the "big ideas" of mathematics and provides students with tools for learning different kinds of contents during their early school years. According to Hurst & Hurrell (2014), however, the nature of students' learning of multiplication in primary and middle school is mostly procedural. The issue of learning multiplication and multiplication tables through empirical learning has also been addressed by researchers such as Gierdien (2009) and Downton (2015). Their empirical studies show that students' learning is basically a matter of memorizing formulas, facts and procedures and they attribute this to culturally based teaching methods. Other studies highlight a focus on multiplication as repeated addition (Askew, 2018; van Dooren, de Bock, & Verschaffel, 2010). According to Fischbein, Deri, Nello, and Marino (1985), students in grades 5, 7 and 9 often intuitively use a primitive model of repeated addition. At the same time, most researchers agree that the structural characteristics of multiplication play an important role in learning the concept of multiplication (Park & Nunes, 2001; Sherin & Fuson, 2005) and how students develop multiplicative thinking (Heng & Sudarshan, 2013). The development of conceptual thinking and its significance for students' multiplicative thinking is described by Wright (2011), who points out that students' previous experiences of applying a concept are crucial for identifying relationships in different contextual situations, thereby activating the students' knowledge as a resource for learning multiplication.

Another theoretical approach to conceptualization and generalization is described by FeldmanHall et al. (2018), who emphasizes that generalization is a logical device usually associated with the process of learning. As a teaching method, generalization is closely associated with the process of "formation" of mathematical concepts as a basis for learning as a mental activity in the transition from perception to concept, e.g., "... a generalization is made – that is, similar qualities in all objects of the same type or class are acknowledged to be general" (Danilov & Esipov, 1957 p. 77). Empirical studies about generalization and conceptualization in the teaching and learning process are described by Kennedy (1997), Onwuegbuzie and Leech (2009) and Williams and Young (2021).

To summarize: students' learning about the concept of multiplication is a complex process that includes an individual's mathematical development within the framework of the generalization process, as well as decisive factors such as which content is actually taught in teaching.

2. BACKGROUND LITERATURE VIEW

The background research for this study includes findings related to generalization and multiplication, mathematical structures and multiplication, as well as guided learning.

2.1. Generalization and Multiplication

A relationship between students' experiences and formal mathematics in terms of concrete-abstract can be based on inner conceptual relationships of mathematics. The transition from concrete to abstract in this process provides aspects such as students' preliminary intuitive knowledge, perceptions and the interaction between them (Hiebert & Behr, 1988; Fennema et.al., 1996). These authors also agree that more research and more knowledge is needed about students' development of conceptually required learning processes.

Empirical studies illustrate that early grade students are able to reason multiplicatively (Steffe, 1994; van Dooren, de Bock, Janssens, & Verschaffel, 2008). Other studies address to students' multiplicative reasoning and thinking, focusing on how students learn multiplication by additive reasoning (Larsson, Pettersson, & Andrews, 2017; Kaufmann, 2018) and by multiplicative reasoning (Sullivan, Clarke, Cheeseman, & Mulligan, 2001; Siemon, Bredd, & Virgona, 2017). Additive and multiplicative reasoning are two different domains for the generalization and conceptualization of multiplication. According to Behr, Harel, Post, and Lesh (1992) students chose additive reasoning intuitively because multiplicative reasoning is more abstract. A transition from an additive to a multiplicative model requires a conceptual change in students' thinking from a linear approach representing just one unit to a rectangular approach presenting two units (Fernandez, Llinares, van Dooren, de Bock, & Verschaffel, 2012).

2.2. Mathematical Structures and Multiplication

The generalization of arithmetic in an algebraic context, especially in the lower grades, has been discussed by other researchers (Mason, 2009; Stephens, Ellis, Blanton, & Brizuela, 2017). Their discussions focused on how an extension of arithmetic in a conceptual sense can be performed. These discussions were followed up by Kieran (2004), who focused on conceptual expansion during students' early years. From a mathematical perspective of view (van der Waerden, 1971), multiplication is an arithmetic operation which, for natural numbers, involves repeated addition and, for other number ranges, is defined by expanding this while preserving the basic laws of algebra. This means that repeated addition only applies when the multipliers are natural numbers. For other number ranges there is a need for adding new conceptual components. Vergnaud (1994), Nunes and Bryant (2010) and Clark and Kamii (1996) describe such multiplication structures. According to Vergnaud (1983), multiplicative structures rely partly on additive structures; but they also have their own intrinsic organization which is not reducible to additive aspects. He defined the concept of multiplication as a relationship between two quantities related to multiplicative situations.

2.3. Guided Learning

Researchers such as Davydov (1990) and Hershkowitz, Schwarz, and Dreyfus (2001) highlight the importance of the guiding principle of instruction. Davydov emphasizes the importance of teachers' focus on the students' perceptions, observations and reflections and their ability to distinguish between essential and crucial conceptual relationships and non-essential relationships. It is also important for the students to verbalize their thoughts and to generalize concepts. Guided learning is a continuous and systematic process that takes place between educators and learners. The key aspect of guiding is "the sensitive, supportive intervention of a teacher in the progress of a learner who is actively involved in some specific tasks, but who is not quite able to manage the task alone" (Mercer, 1995, p. 74).

The present study does not attempt to investigate an epistemological context related to the generalization of multiplication by multiplicative structures, such as different multiplication models. The study is based on a socio-cultural paradigm and examines students' learning of multiplication by generalizing the concept of multiplication, using multiplicative structures and the basic laws of algebra, and how to apply this to understanding and systematizing patterns in the multiplication tables according to Davydov's (1990) and Vergnaud's (1983) theoretical approach. The research questions are:

(RQ1) How do students visually express their conceptual generalizations (grade 3)?

(RQ2) How do students apply their conceptualization of representations of multiplication, including segments such as *identifying*, *classifying* and *systematizing structures of multiplication*, to solving multiplication table tasks (grade 5)?

3. THEORETICAL FRAMEWORK AND DESIGN

Davydov's (1990) model emphasizes the connection between generalization and conceptualization. According to Davydov, generalization – a phenomenon related to mental processes – is used to describe different aspects of students' learning. The empirical-theoretical approach adopted by Davydov indicates that generalization and conceptualization are key components in teaching school mathematics. A consequence of generalization is that students' understanding of the nature of facts can be expressed verbally and recognized in a familiar setting.

Mathematical activities in teaching are necessary prerequisites for developing the ability of students to generalize and conceptualize. Such activities must be planned with emphasis on the content of concepts and the teacher-student discourse, in which students analyze, constitute, recognize and produce verbal responses. Davydov interprets the generalization process as comprising three linked elements: perception, conception and concept. Students' apprehension of a concept is attributable to generalized perceptions and conceptions of many similar objects with a focus on the crucial properties of the objects. The transition from perception to concept is not an easy process to grasp for primary school students. For them, generalization is a form of representation of "elementary concepts". For the purposes of this study, the model of generalization used is the "perception-conception-elementary concept" (PCE).

According to Davydov, the transition from perception to concept takes place through different forms of visualization: *symbolic* (graphic, drawing), *verbal* (explanation of different situations, stories), *natural* (everyday situations, physical objects) and *artificial* (classroom discourse and context). Attributes for students' ability to generalize comprise being familiar with concepts and their crucial properties and *applying* the concepts in practice and in different contexts to generate *new knowledge.* In the primary grades, generalization as a transition from perception to concept is related to the *representations of elementary concepts* by students' ability to *identify*, *classify* and *systematize* them. According to Vergnaud (1983), different multiplicative problems can be described by different multiplication models. These models can be divided into three classes as a basis for multiplicative structures: the *mapping rule* (MR), also known as *repeated addition*; *multiplicative comparison* (MC), also known as *enlargement*; and *Cartesian product* (CP).

In order to extend the mathematical content of multiplication, the algebraic laws of multiplication were taken into account (van der Waerden, 1971). The commutative, associative and distributive laws, in interaction with multiplicative structures, enabled the discovery of different dimensions of the concept of multiplication and crucial patterns in the multiplication tables. Davydov's theory (1990) of the generalization of mathematical concepts constitutes the design of the study and Vergnaud's (1983) and van der Waerden's theoretical models for the content are the focus of this study.

4. METHOD

4.1. Variation Theory

Variation theory (Marton, 2015), with its roots in phenomenography, is a general theory of learning. For learning to take place, there must be a focus on the crucial aspects of the "*object of learning*". There must also be a degree of variation in some of these crucial aspects, while other aspects remain constant. From the teacher's perspective, this requires a good overview of and insight into the content of the object, as well as knowledge of the subject in question (Shulman, 1986), otherwise the teacher will neither be able to plan sustainable teaching nor identify crucial aspects of students' conceptions of the actual phenomenon. In this study, variation theory is used as a methodological design in order to qualitatively analyze the wide range of conceptual properties for multiplication in students' learning. This is related to the teaching process and activities, emphasizing multiplicative structures, the basic laws of multiplication, and how students apply this to multiplication situations related to patterns in the multiplication tables.

4.2. One-on-one Interviews

4.2.1. Interview Structure

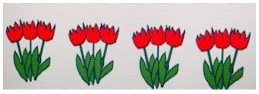
During the interviews, 12 different questions were asked about multiplicative situations (see Table 1 and Table 2).

Table 1.
Interview questions, part 1.

Multiplication

MR structure

(1) How many flowers are there in the picture? Describe this (a) as addition.
(b) as multiplication.



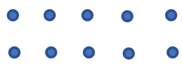
(2) Write as multiplication: (a) 5+5+5+5+5=__; (b) 4+4+4+4+4+4+4=__

MC structure

(3) A pear costs eight kronor and an apple costs ten kronor. (a) How much do five pears cost?
(b) How much do five apples and one pear cost?

CP structure

(4) Describe and count by multiplication how many dots there are in the following picture:



(5) Describe in the same way using a picture: (a) the multiplication 6·7. (b) the multiplication 3·7.

Representation of multiplication

(6) Draw a picture of the multiplication 5·3.

Table 2.
Interview questions, part 2.

The basic laws of algebra

The commutative law

(7) Which sum is greater: 7+7+7+7 or 4+4+4+4+4+4+4? Explain how you worked this out.

The distributive law

(8) The circumference of a rectangle is 7+4+7+4 centimeters. Choose the correct answers:
The circumference is (a) 4·7 centimeters, (b) 2·7+2·4 centimeters, (c) 2·(7+4) centimeters.

Multiplication tables

(9) Show how it is possible to find 8+8+8 in the multiplication table.
(10) Look for 3·8 and 8·3 in the multiplication table. What did you find?
(11) Show how it is possible to find 8·6 in the multiplication table if you know that 8·5=40.
(12) Where are the even and odd numbers in the multiplication table?

The interview questions have been developed from Vergnaud's theory of multiplicative structures: the mapping rule (MR), multiplicative comparison (MC) and the

Cartesian product (CP), and from van der Waerden's (1971) definition of the basic laws of algebra but were extended to incorporate different multiplicative problem situations. Regarding *Table 2 (Part 2)* the students received support and guidance during the interviews. The same questions were used during the interviews in grade 5 as in grade 3, in order to identify conceptual developments in students' learning.

4.3. Data Collection

The study participants comprised two teachers and 40 students in two grade 3 classes, followed up two years later by 25 of the same students, who were now in grade 5. Data were collected by observing the teaching process and conducting one-on-one interviews with students in both grade 3 and grade 5. All data were transcribed and systematized for analysis related to the theoretical tools.

5. DATA ANALYSIS

In order to answer research questions RQ1 and RQ2, a two-level qualitative analysis was conducted. The quantitative data from the interviews with the students is only intended to be a background for reflecting on the fundamental differences between the qualitative and the quantitative data. The aim of the analysis was to understand the qualitative dimensions of students' learning regarding perceptions and representation of the concept of multiplication.

6. FINDINGS OF THE STUDY

6.1. Observations

Finding 1

During one lesson (T1) in grade 3, the teaching activity was based on the picture in Figure 1. The task was formulated thus: *Describe the picture as addition and as multiplication.*

Figure 1.
Picture from the textbook Favorite Mathematics (2013).

The students wrote their answers on whiteboards. When the students showed their answers, the teacher then asked them if the various answers were right or wrong.

Teacher: All of you have written 3+3+3+3. Let's write the sum 12.
....
Teacher: *As a multiplication we have 4·3*. [Writes 4·3 =12]
They then discuss the answer 3·12=12 and found that it was wrong.
Teacher: *Now we have 3· 4=12. Earlier we had 4·3=12. What do you think?*
Student 1: *There are 4 groups and 3 in each group.*
Teacher: *Do you agree?*
Student 2: *Yes, there are 3 groups and 4 in each group.* (Wrong!)
Teacher: *Do you agree?*
Students: *Yes.*

....
Teacher: *Finally, we have 2· 6= 12.* (Correct)
Students: *Wrong.*
Teacher: *Yes.*

The object of learning was addition as an introduction to multiplication. The answers show the students' varying perceptions, as described in their explanations.

Finding 2

During another lesson (T2) in grade 3, multiplication as repeated addition was introduced using a picture of 10 circles in a 2 x 5 pattern.

Teacher: *What kind of multiplication fits this picture?*
Students: [Silence. No response]
Teacher: *Multiplication?*
Student 1: *25.*
Teacher: *Are there 25 circles here on the board? No, there are not 25 circles. How many rows of circles are there?*
Student 2: *2*
Teacher: [Writes 2] *How many circles are there in each row*?
Student 3: *5*
Teacher: *5.* [Writes times 5] *And how many circles will there be? 2 times 5 equals…*
Student 3: *10*
Teacher: *Yes, and now let's count the circles: 1, 2, 3, 4, 5, 6, 7, 8, 9,10, so it can't be 25. I can only find 10 circles.*

The object of learning was repeated addition. A logical approach had been to start with 2 + 2 + 2 + 2 + 2 and express this as 5 · 2 circles. In the same way, the rows contain 5 + 5 circles, which can be expressed as 2 · 5 circles. Student 1 explained by focusing on 5 and 5 and used multiplication and student 3 gave a correct answer. Provocative was that teacher instead of explaining 25 as 5 + 5, counted the circles one by one. The researchers paid attention to how the teacher lost track and focused on proving that there were just 10 circles.

6.2. Interviews

The interviews in grade 3 show that most students were interested and happy *to describe* and *explain* the various multiplicative situations. However, the students' perceptions and experience of multiplication had limitations regarding the concept of multiplication. (Figure 2). This became obvious in question 6: *Draw a picture of the multiplication* 5· 3. Only every second student gave the correct answer (Table 3).

Table 3.
Students' answers to question 6.

Answer	15	18	20	No answer
Number of answers	20	10	1	9

Figure 2.
Examples of how students described the multiplication 5·3 using pictures.

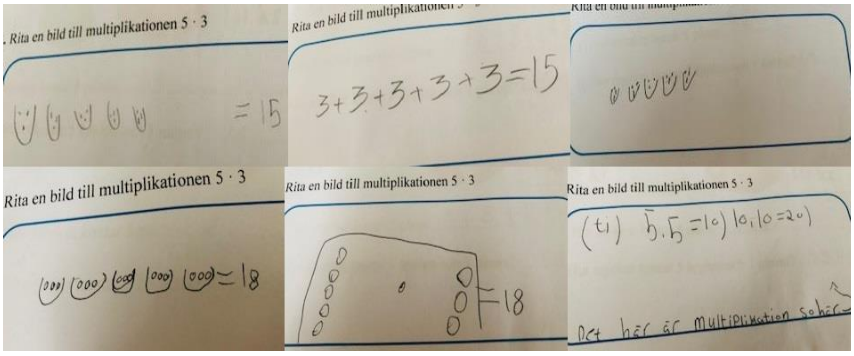


Figure 2 shows that the students drew different representations of multiplication. Dominant was repeated addition and equals groups. The students also described multiplication using calculations (5·3=18 and 5·5=10...). During the interviews, the researchers guided the students with follow-up questions and discussions. At the end of the interviews, all students were able to describe both multiplication and repeated addition as equal groups. However, the students found it difficult to identify the structure of multiplication as a rectangular structure and its commutativity. It was also difficult to understand Cartesian structure and commutativity.

The interviews in grade 5 showed that only a few students were able to identify the property of commutativity. During the interviews, one of the questions was: *Which sum is greater, 4+4+4+4+4+4+4 or 7+7+7+7*? *Explain how you worked this out* (Table 2, question 7).

Student 1: *It'*s *7+7+7+7*
Interviewer: *Why, can you explain?*
Student 1: *Because 7 is bigger than 4*

Student 2: *I think it's the one with the 4s*
Interviewer: *Can you explain why?*
Student 2: *4+4 equals 8. And there are just four 7s. So I think this one is bigger.*

The students' explanations were based on a comparison of two natural numbers, 4 and 7.

During the interviews the students' got problems when they had to analyze patterns in 3 times, 5 times, 6 times and 8 times tables (see Table 2, questions 9 to 12). An interesting observation was that just a few of the grade 5 students knew the multiplication tables by heart and just one of them was able to identify a repeated addition in the multiplication table (see Table 2, question 9, MR structure), the commutativity in question 10, and to describe multiplication using enlargement (see Table 2, question 11, MC structure). This showed the importance of the students' perceptions and the development of MR into MC and CP. The students' interpretation of multiplicative situations illustrated that their representations of multiplication were limited. During the interviews time, with careful guiding from researchers, the situation was different. Now, all the students were able to understand the different structures in the multiplication table. However, it was still difficult for them to change from repeated addition (MR structure) to Cartesian representation (CP structure) in the multiplication table.

7. CONCLUSION AND DISCUSSION

RQ1: How do students visually express their conceptual generalizations?

This study shows that a generalization of multiplication, particularly of concepts such as MR to MC and CP and the basic laws of algebra, is a difficult process for students to grasp. The students' perceptions of multiplication in grade 3 varied significantly and they expressed themselves using various explanation, although there were limited possibilities for further development. The results in grade 5 show that the generalization of multiplication from MR to CP plays a key role in students' perception of multiplication. It should be noted that the students' perception is the first level of the PCE model, and if this level is not achieved, generalization of the students' learning of the concept of multiplication, and their learning as a process, will not occur (Davydov, 1990).

It should be noted that the students' representations of multiplication in grade 5 were only performed as equal groups or repeated addition, although similar tasks were discussed in detail, as early as grade 3. The students' difficulties with the CP structure are explained by Behr et al. (1992), who stated that students chose additive reasoning intuitively because multiplicative reasoning is more abstract. This study shows that students are still more confident with the MR structure in grade 5 and that teaching ought to provide more effective ways for students to develop multiplication (Mason, 2009; Stephens et al., 2017). All tasks in grade 3 invited students to develop multiplication, but with no conceptual support from their teachers, and a suitable variation (Marton, 2015). It was difficult for the students to shift between an additive and a multiplicative model (Askew, 2018). This demonstrated that it is also important for teachers to use guided learning (Davydov, 1990; Hershkowitz et al., 2001) and mathematical communication with the clear object of learning and a structured and relevant content (Mercer, 1995). The teacher's role is to support and guide students in generalizing multiplication and perceiving abstract multiplicative structures.

RQ2: How do students apply their conceptualization of representations of multiplication, including segments such as identifying, classifying and systematizing structures of multiplication, to solving multiplication tables tasks?

This study shows that students' conceptualizations of multiplication were under development. Most of them were able to identify, classify and make representations of multiplication using an MR structure such as repeated addition and equal groups in grades 3 and 5 (Fernandez et. al., 2012). At the beginning of the interviews in grade 5, few students were able to identify CP structures, use them to analyze multiplication and identify patterns in the multiplication tables, not even the four students who knew the multiplication table by heart. At the end of the interviews, most of the students were able to discuss multiplicative situations and read the multiplication table using MR and MC structures (Kieran, 2004). They were also able to identify the relationships between even and odd products, commutativity and symmetry in the table, and sometimes even patterns in the 5- and 9-times tables. However, unless such aspects are highlighted in teaching, students will not be aware of them. The examples from the interviews show that a combination of challenging tasks and conceptual support encourages students to develop new ways of thinking.

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