Chapter #8

LEARNING NON-EUCLIDEAN GEOMETRIES: IMPACT EVALUATION ON ITALIAN HIGH-SCHOOL STUDENTS REGARDING THE GEOMETRIC THINKING ACCORDING TO THE VAN HIELE THEORY

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ABSTRACT
This paper aims to explore the impact of a non-Euclidean geometry course on Italian high-school students regarding the assessmentof geometric thinking. To accomplish this, we analyse the results of the van Hiele levels test. We slightly modified and translated to Italian the van Hiele test, originally designed by Usiskin, and we used it to detect possible changes of the students' levels of geometric thinking after we taught a non-Euclidean geometries course of our design. The students involved in the test (N=56) span ages 15-18 and all attend the "Liceo Scientifico" high school type. The results show that there is a statistically significant (p-value < 0.05) improvement in the median level of understanding in geometry if we consider the so-called modified van Hiele theory. Since we observe this improvement only for classes with an entering van Hiele level of at least 3, we suggest our non-Euclidean geometry course only for these classes of students, regardless of the grade.

Keywords: non-Euclidean geometries, geometric thinking, van Hiele theory, high school.

1. INTRODUCTION

Geometry is an essential part of the education of secondary school students and it usually focuses on the Euclidean geometry. The study presented hereby is part of a broader project on the teaching and learning of non-Euclidean geometry at Italian high school (Benvenuti & Cardinali, 2018) (Cardinali & Benvenuti, 2020) which deals specifically with "Liceo Scientifico", one of the most common and long-lived high-school types. Such curriculum provides the student, along with several human-related subjects, a broad vision of past and present human knowledge. Specifically, it provides analytic means to shape the student's ability to understand scientific knowledge, the progress of scientific thought and all the tools to develop an open mind for scientific studies if enrolled in a university program. Current "Liceo Scientifico" national recommendations introduced in 2010 (MIUR, 1996) suggest that the student should be able – at the end of her/his studies – to understand the historical context of several mathematical theories and their conceptual meaning. The recommendations also suggest "a clear vision of the axiomatic approach in its modern conception and of its specificity with respect to the classic Euclidean approach". To achieve this goal, we believe that the teaching of non-Euclidean geometry is useful, if not indispensable. In fact, as stated in (Magnani, 1978), non-Euclidean geometries are a fundamental link in the transition from classical to modern Mathematics. Furthermore, concerning non-Euclidean geometry, Toth writes that without it, the development of so-called modern mathematics would be hardly conceivable (Toth, 2003), Zheng discusses how it bears on problems of the nature of mathematical truth and modes of thought (Zheng, 1992), Kline refers to its creation as "the most consequential and revolutionary step in mathematics since Greek times" (Kline, 1972, p. 879). Despite this, the current Italian recommendations do not include teaching of non-Euclidean geometry among the suggested topics. For this reason, the study we have conducted on the impact of an elementary non-Euclidean geometries course for high-school students is of an experimental type.

The goal of this work is to assess the impact of a short non-Euclidean geometry course on the development of geometric thinking according to the van Hiele theory. To reach that goal, we designed an elementary non-Euclidean course which we provided to high-school students, and we evaluated its impact by means of the van Hiele test formulated by Usiskin in (Usiskin, 1982). The test was translated to Italian and, as a slight modification to the original test, we clarified in the items, when necessary, that the question was referred to a Euclidean plane. The test was provided before and after the course, to assess the initial van Hiele geometry levels of the students, and to compare these with the final ones.

1.1. The van Hiele Model

The van Hiele model is a theory that describes how students reason, when solving geometrical problems or working with geometrical elements (e.g. definitions, classifications). A husband-and-wife team of educators, Pierre van Hiele and Dina van Hiele-Geldof, developed it in their thesis at the University of Utrecht in 1957 (Usiskin, 1982). They postulated five levels of thought in geometry, each level indicates how individuals think over geometrical concepts. Hoffer summarizes – and Usiskin proposes again – general descriptions of the van Hiele's levels as follows (Usiskin, 1982) (Hoffer, 1979) (Hoffer, 1981):

Level 1 (recognition): the student can learn names of figures and recognizes a shape as a whole (e.g.: squares and rectangles seem to be different).

Level 2 (analysis): the student can identify properties of figures (e.g.: rectangles have four right angles).

Level 3 (order): the student can logically order figures and relationships but does not operate within a mathematical system (e.g.: simple deduction can be followed, but proof is not understood).

Level 4 (deduction): the student understands the significance of deduction and the roles of postulates, theorems, and proof (e.g.: proofs can be written with understanding).

Level 5 (rigor): the student understands the necessity for rigor and can make abstract deductions (e.g.: non-Euclidean geometry can be understood).

Like Usiskin, we point out that van Hiele number these levels 0 through 4, not 1 through 5. Moreover, Dina van Hiele-Geldof associates different names to the levels with respect to the levels 2 through 5 indicated above: "the aspect of geometry" (level 2), "the essence of geometry" (level 3), "insight into the theory of geometry" (level 4), and "scientific insight into geometry" (level 5) (Hiele-Geldof, 1957).

Pierre M. van Hiele identifies four properties of the levels (Van Hiele, 1958-59), to which Usiskin assigned names (Usiskin, 1982):

Property 1 (fixed sequence): a student cannot be at van Hiele level n without having gone through level $n–1$.

Property 2 (adjacency): at each level of thought what was intrinsic in the preceding level becomes extrinsic in the current level.

Property 3 (distinction): each level has its own linguistic symbols and its own network of relationships connecting those symbols.

Property 4 (separation): two persons who reason at different levels cannot understand each other.

The van Hiele theory explains – and Usiskin's study confirms – why many students have troubles learning and performing in geometry classes: the weak performances of many students are often associated with being at a lower van Hiele level with respect to the level of the teaching.

1.2. Usiskin's van Hiele test

Usiskin designed a test ("van Hiele test") to detect the level of thought in geometry according to the van Hiele theory (Usiskin, 1982). There are 25 questions, 5 questions for each level.

There are two criteria to assess if a student satisfies a certain level ("fits"): the "3 of 5 criterion" and the "4 of 5 criterion". The first one considers the level as passed if the student answers correctly to at least 3 of the 5 questions of that level. The second one, called "strict criterion", considers the level passed only if the student answers correctly to at least 4 of the 5 questions of that level. Usiskin suggests that the choice of the criterion is done based on the wish to reduce Type I (false positive) or Type II error (false negative).

Usiskin observed that sometimes level 5 items turned out to be easier for students than items at levels 4 or even 3, and that the reliability of the test for the fifth level is discussed. For these reasons, Usiskin considers two different theories: the classical one and the modified one. The so-called modified theory differs from the classical one for the fact that level 5 is not considered. Thus, for example, if a student satisfies (according to a certain criterion) level 1, 2, 3, and 5 (but not level 4), he/she is classifiable only under the modified theory. Specifically, the student fits level 3 of the modified theory. The assigning of levels in either the classical or modified case requires that the student at level *n* satisfy the criterion not only at that level but also at all preceding levels (Usiskin, 1982).

2. MATERIALS AND METHODS

2.1. Participants

Currently, non-Euclidean geometries are not considered by Italian high-school guidelines, therefore their teaching is not compulsory. To ensure a collaboration with teachers, we proposed our course to high-school teachers that had already expressed their interest in conducting a course in non-Euclidean geometry.

The high schools involved were two "Liceo Scientifico" high school type. Several curricula of "Liceo Scientifico" exist. One of the four classes that took part in the study follows the traditional curriculum of the "Liceo Scientifico" already discussed in the introduction, two classes are involved in the "Scienze Applicate" (Applied Sciences) curriculum, while another one is involved in the "Cambridge International" curriculum. "Scienze Applicate" curriculum renounces some of the aspects of humanistic culture, those linked to the study of Latin classicism, in favor of more scientific oriented programmes. The "Cambridge International" curriculum allows to learn the English language at high levels of competence by supporting the English teacher with a native speaker, and teaching two disciplines, generally of a scientific nature, in two languages. In their first two years all the curricula of "Liceo Scientifico" deal with the study of Euclidean geometry from the axiomatic point of view.

Students from two classes for each school took part in the project. Specifically, 18 students from a second-grade class, 25 students from a third-grade class, and two sets of students from two fifth-grade classes (a set of 10 students and the other set made of 24 students). No one of the students had learning disabilities identified.

Table 1.
Subjects involved in the study.

	Set 1 (II SA class)	**Set 2 (III CI class)**	**Set 3 (V SA class)**	**Set 4 (V LS class)**
High school	Liceo Scientifico School A	Liceo Scientifico School A	Liceo Scientifico School B	Liceo Scientifico School B
Curriculum	“Scienze Applicate”	“Cambridge International”	“Scienze Applicate”	No special curriculum
Students	Students from a single II grade class	Students from a single III grade class	Students from a single V grade class	Students from a single V grade class
Number of students who attended the course	18	25	10	24
Number of subjects *	14	20	8	14

*Students who attended the course and answered all the questionnaires

The inconveniences created by the Covid-19 pandemic reduced the number of classes involved in the study and forced us to conduct the course outside school hours. To avoid dispersion, the teachers strongly encouraged their students to attend the course and demanded from them to justify their eventual inability to participate (i.e. students who practice sport at a competitive level have mandatory afternoon workouts and have therefore been justified).

Table 1 shows data on the subjects involved in the study. Note that the number of subjects involved is minor that the total number of students who attended the course. This because we consider as subject of our study only those students who attended the course and answered not only to the van Hiele test but to all the questionnaires planned for our main research.

2.2. Class activities

The course consisted of five two-hours sessions, a session every week. It was conducted between mid-October and the end of November 2020 (during the Covid-19 pandemic), after school hours, by the first author. The restrictions imposed by the anti-pandemic plan forced us to conduct the course in online mode (we used the software “Cisco Webex Meetings”). We chose to conduct the course in synchronous mode because all students involved in our study were used to this lesson mode. The alternative could have been to conduct asynchronously a part of the course, using podcasts, with the flipped classroom method. We did not choose this method because it was not a well-known and common practice among all groups of students. Moreover, the synchronous mode allowed us to have immediate feedbacks on the topics covered from the students. Nevertheless, it would be interesting to re-propose our study using the flipped classroom mode. The better mode to conduct our course – we believe – is the one in presence. In this case, we also suggest conducting the activity on the Poincaré disk model in group work mode.

Before starting the course, we ensured that all students had the necessary materials for the workshops. Among these, the following material: polystyrene hemispheres that can be written with markers or pinned; sewing cotton and pins to draw straight lines on the polystyrene hemispheres; rulers; protractor; and compass. The teacher (the first author) was provided with the same materials as the students and more: a globe; a tiny toy car with no steering; 3D-printed hemispheres, pseudospheres and flexible ruler. All the material used in the workshops is shown in Figure 1.

Figure 1.
Some material used for class activities.

The activities proposed to the students were planned as shown in Table 2. Five meetings of two hours each, beginning with an interactive session whose main objective was to understand what a circle and a straight line look like on a spherical surface, during this session we also deal with the definitions of segment, angle, polygon, and triangle on a spherical surface. At the beginning of the second session, the teacher of the course assigned the students to groups based on their results at the pre-test, in such a way as to minimize the chance that high ability students will huddle together leaving others out. Each group was of four students, exception for some groups of three students. The teacher also created a virtual room for each group on the Cisco Webex platform, virtual rooms in which she could log into to monitor the work of the groups. During the second session, the students, divided into groups, tried their hand at tasks to be carried out on polystyrene hemispheres. These tasks allowed each student to explore the spherical surface and to observe that there exist geometric figures' properties that hold on a plane surface while they do not hold on a spherical surface. We deduced that we were dealing with a geometry different from the one we already knew (the Euclidean geometry). The third session revolved around the following question: "Why, are there geometric figures' properties that hold on a plane surface and that does not hold on a spherical surface, and vice versa?". We refreshed the basic elements of the Euclidean geometry and discussed on the eventual validity of the five postulates of Euclid on a spherical surface. We observed that there are interpretations that allow us to consider the five postulates, formulated by Euclid, also valid in spherical geometry (Carroll & Rykken, 2018). Afterward, we analysed the Proposition 31 of the first Book of the Euclid's *Elements* ("Through a given point to draw a straight line parallel to a given straight line"), and its proof that relies on Proposition 16 (*Exterior Angle Theorem*). We understood that there is a flaw in the proof of Proposition 31. This led the teacher to mention the Hilbert formalization of Euclidean geometry (specifically, the third axiom of order and the axiom of parallel), and the meaning of consistency, completeness, and independence of an axiomatic system. Connecting to the concept of independence of an axiomatic system, the fourth session focused on the controversy surrounding Euclid's fifth postulate, on the birth of the hyperbolic geometry, and on the importance of having models for an axiomatic system. The teacher used a 3D-printing models of pseudospheres to show geometric figures' properties that hold on a plane surface but that does not hold on a pseudosphere and vice versa. The fourth session ended with a discussion on the loss of meaning of the question "Which geometry is the true one?", and contextualizing non-Euclidean geometries from an application point of view (linking e.g. to relativity in physics or the global positioning system in engineering). Finally, the last meeting consisted of a workshop on the Poincaré disk model and on a final discussion to resume the all course. The aim of the workshop on the Poincaré disk model was to let the students become more familiar with hyperbolic geometry, understand that there can be more than a model for a geometry, and avoid the misconception of identifying a geometry with one of its models.

Table 2.
Plan of the class activities.

	Topic	**Working format (online)**
I session (2 hours)	Circle, straight lines, segments, angles, polygons on a spherical surface	Frontal-dialogue lesson/ Workshop
II session (2 hours)	Constructions on a spherical surface	Group work
III session (2 hours)	Euclidean geometry and the eventual validity on a spherical surface of the five postulates formulated by Euclid	Frontal-dialogue lesson
	Euclid's flaw on Proposition I.16 and mention to the Hilbert formalization of Euclidean geometry	
	Introduction to the meaning of consistency, completeness, and independence of an axiomatic system	
IV session (2 hours)	The independence of the fifth postulate of Euclid	Frontal-dialogue lesson
	Hyperbolic geometry	
	Models for an axiomatic system	
	"Which geometry is the true one?"	
V session (2 hours)	Poincaré disk model	Workshop with "NonEuclid" software
	Final discussion	Frontal-dialogue lesson

2.3. Data collection

As discussed above, we used the van Hiele test to assesses the students' levels of geometric thinking according to the van Hiele theory. The test was translated to Italian and one clarification was added to some items. Specifically, some questions now clarify that they refer to the Euclidean plane, to avoid confusion with other non-Euclidean surfaces that the students encountered in our course.

We distributed the tests (pre-test and post-test) via Google Form. We clarified with the students that: only the researchers involved in the study would see their answers; there would be no evaluation; the researchers involved in the study could contact them to discuss their answers.

3. RESULTS

In the present section, we report results regarding the van Hiele levels detected by the van Hiele test before and after the class activities on non-Euclidean geometries, Since, to the best of our knowledge, there are no studies similar to ours, we do not have a comparative term for the effect sizes of our interventions. Nevertheless, we will report the effect sizes because it could be useful for future studies. Indeed, like observed in (Bakker, Cai, English, Kaiser, & Mesa, 2019), findings on effect size should be related to "comparable studies with similar characteristics (research design, sample size, type of measurement, type of variable influenced, etc.)" in terms of "smaller/larger than typical under such conditions," or "comparable with other studies with similar characteristics (research design, alignment between intervention and assessment, sample size, type of variable influenced etc.)".In the next subsection we report results that answer the following question:

> Q1. How are students distributed before the class activities on non-Euclidean geometries with respect to the levels detected by the van Hiele test?
>
> Q2. How are students distributed after the class activities on non-Euclidean geometries with respect to the levels detected by the van Hiele test?

We answer the two previous questions considering: case 1) all the 56 students who answered to all the four questionnaires involved by the experimentation, before and after the course; case 2) only the students who fit the classical van Hiele theory both in the pre-test and in the both-test; and case 3) only the students who fit the modified van Hiele theory both in the pre-test and in the both-test. Each of the previous three cases are divided in two subcases: the 3 of 5 criterion and the 4 of 5 criterion. For cases 2) and 3) we state if the differences between the post-test and the pre-test are significative and we report the effect sizes of the non-Euclidean activities on the levels detected by the van Hiele test. For case 1 we cannot report the effect size or whether the difference is significative because there are students that do not fit any van Hiele level in the pre-test or in the post-test or in both the tests.

As stated in (Usiskin, 1969) regarding his van Hiele test, for what concern the reliability, the van Hiele test is considered as 5-item tests. The computed Cronbach's α for the five parts in the pre-test are 0.44, 0.54, 0.56, -0.13, and 0.67, while in the post-test the computed Cronbach's α are 0.58, 0.61, 0.78, 0.52, and 0.39. We observe, as done by Usiskin, that one reason for the low reliabilities is the small number of items; similar tests at each level 20 items long would have the following Cronbach's α: 0.89, 0.91, 0.92, 0.79, and 0.94 in the pre-test, while 0.92, 0.92, 0.96, 0.90, and 0.91 in the post-test.

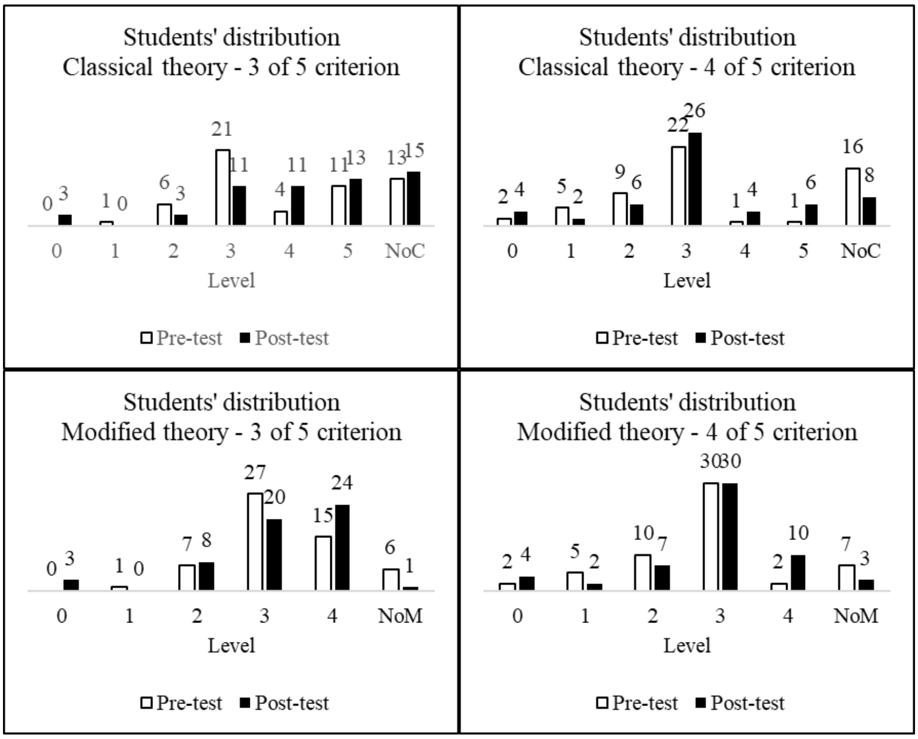
3.1. Results regarding students who answered to all the questionnaires foreseen by the experimentation (case 1)

The graphs in Figure 2 show the students' distribution with respect to the levels detected by the van Hiele test. All the 56 students who answered to all the questionnaires foreseen by the experimentation are included. Analysing the pre-test, we see that – according to the 3 of 5 criterion and the 4 of 5 criterion, respectively – roughly 23% and 27% of students do not fit the classical theory, while roughly 11% and 12% of students do not fit the modified theory. Analysing the post-test, we see that – according to the 3 of 5 criterion and the 4 of 5 criterion, respectively – roughly 23% and 27% of students do not fit the classical van Hiele level, while roughly the 21% and 4% of students do not fit the modified theory.

3.2. Results regarding students who fit the classical theory both in the pre-test and in the post-test (case 2)

We answer question Q1 and question Q2 written at the beginning of the present section considering only students who, according to Usiskin (Usiskin, 1982), fit the classical theory, both in the pre-test and in the post-test.

Figure 2.
Distribution of the 56 students with respect to the levels detected by the van Hiele.
(a) Classical theory - 3 of 5 criterion; (b) Classical theory - 4 of 5 criterion;
(c) Modified theory - 3 of 5 criterion; (d) Modified theory - 4 of 5 criterion.



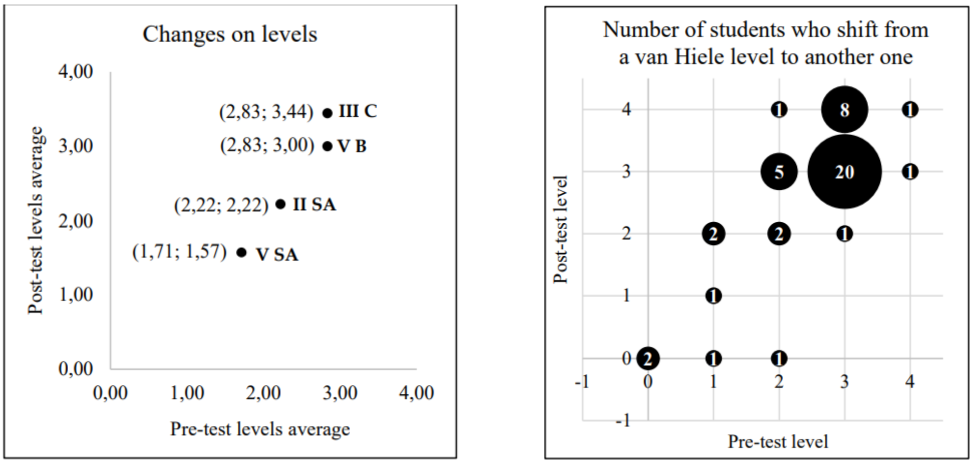
For the 3 of 5 criterion, the mean levels are 3.45 (standard deviation: 1.18) and 3.76 (standard deviation: 1.35), respectively in the pre-test and in the post test. For the 4 of 5 criterion, the mean levels are 2.43 (standard deviation: 1.01) and 2.71 (standard deviation: 1.25), respectively in the pre-test and in the post test. In both cases (3 of 5 criterion and 4 of 5 criterion), the levels for the post-test seem to be, on average, slightly higher than the ones for the pre-test but the difference is not statistically significant. Indeed, we conducted the Wilcoxon Signed-Rank test to understand whether reject the following hypothesis h_0: "the median level before and after the workshop is identical" and its results does not allow us to reject the null hypothesis with a high confidence (p-value > 0.05). We also computed the effect size for each case: the Cohen's d computed value is 0.24 in the case of the 3 of 5 criterion, while it is 0.25 in the case of the 4 of 5 criterion.

3.3. Results regarding students who fit the modified theory both in the pre-test and in the post-test (case 3)

We answer question Q1 and question Q2 written at the beginning of the present section considering only students who, according to Usiskin (Usiskin, 1982), fit the modified theory (i.e., the theory if level 5 is removed from consideration), in both the pre-test and the post-test.

Figure 3.

(a) Changes on the average levels for each group of students (modified theory - 4 of 5 criterion); (b) Number of students who shift from a van Hiele level in the pre-test to another one in the post-test (modified theory - 4 of 5 criterion).



For the 3 of 5 criterion, the mean levels are 3.12 (standard deviation: 0.73) and 3.22 (standard deviation: 0.98), respectively in the pre-test and in the post test. The levels for the post-test seem to be, on average, slightly higher than the one for the pre-test but the difference is not statistically significant: the Wilcoxon Signed-Rank test we conducted does not allow us to reject the null hypothesis h_0 (h_0: "the median level before and after the workshop is identical") with a high confidence as p-value < 0.05 (p-value = 0.36). The Cohen's *d effect size* we computed is 0.12.

For the 4 of 5 criterion, the mean levels are 2.54 (standard deviation: 0.89) and 2.80 (standard deviation 1.09), respectively in the pre-test and in the post test. The level for the post-test is, on average, higher than the one for the pre-test and the Wilcoxon Signed-Rank test we conducted let us conclude the difference is statistically significant (p-value < 0.05) so we can reject the null hypothesis h_0 (h_0: “the median level before and after the workshop is identical”) with a high confidence as p-value < 0.05: p-value = 0.03. In this case, the Cohen’s *d effect size* is 0.26.

Since the changes in last case (modified theory - 4 of 5 criterion) are the ones with the highest effect size and since changes are statistically significant, we give some more quantitative details on it.

We show in Figure 3a the changes between the average levels resulted from the pre-test and the ones from the post-test for each set of students (II SA: 9 students; III CI: 18 students; V SA: 7 students; V SC: 12 students). Changes are positive only for those classes whose starting level is roughly 3, these classes are V SC and III CI. The Wilcoxon Signed-Rank test we conducted let us conclude the change regarding set III C is statistically significant (p-value < 0.01). Set V SA worsen it result while group II SA does not change its average. We can conclude that changes do not depend on the grade but on the starting level of thought in geometry.

Figure 3b shows in detail how many students improve, worsen, or do not change their level of thought in geometry (according to the modified theory – 4 of 5 criterion). About the 34,8% of students improves, about the 8.7% of students worsens and about the 65.2% of students does not change their level.

4. DISCUSSION

From the gathered results we observe (Figure 3a), that changes do not depend on the grade whereas it depends on the starting level of thought in geometry. This should let us conclude that the class activities on non-Euclidean geometries - at least in the way we designed them - should be conducted after having tested students’ level of thought in geometry. Non-Euclidean geometry seminars in Italy are often conducted with V grade students (the last grade before university), assuming that their level of thought is high enough to learn non-Euclidean geometries. However, this seems to be not necessarily the case. On the other hand, we have seen that the set of students from III grade class involved in our study have sufficient abstraction level to learn basic concept of non-Euclidean geometries and correctly express concepts of axiomatic geometry. Considering this, we may recommend conducting non-Euclidean geometry seminars only to classes that have a high overall geometry thought, at least 3 according to the van Hiele test. This may reduce the applicability of our method since, previous study show that many students do not reach level 3. As an example, in a recent Czech study (Haviger & Vojkůvková, 2015), the number of students reaching level 3 on a sample of 215 students was 39%. The same result was achieved by Usiskin (Usiskin, 1982) in the USA. It must be noticed that our study is aimed at “Liceo Scientifico” high school, therefore our data cannot be directly compared to the Czech study, where three types of high school are addressed but no disaggregate data is given.

5. CONCLUSION

We hereby reported results from an experimental evaluation of the impact of a non-Euclidean geometry course for different classes and starting with different knowledge levels. We described the method, which was adapted to synchronous online teaching due to the restrictions imposed by the anti-pandemic plan and discussed the changes on the student’s levels of thought in geometry. We detected these changes analysing the van Hiele test filled out by the students before and after the non-Euclidean geometries course. Results were reported, showing that the course had an impact depending on the students' abilities, rather than their grade. Assessing their level with the van Hiele test is, therefore, a necessary step, should this course – or similar other non-Euclidean geometry courses – be taught in school.

The extent of our analysis has been somewhat limited by the rise of the Covid-19 pandemic. The inconveniences it created forced some teachers, who already got engaged in the study, to give up on participating. Therefore, the number of subjects involved in our study dropped dramatically. The results collected in this work should be extended in the future by repeating the experiments on a larger statistical base.

REFERENCES

Bakker, A., Cai, J., English, L., Kaiser, G., & Mesa, W. (2019). Beyond small, medium, or large: points of consideration when interpreting effect sizes. *Educational Studies in Mathematics, 102(1), 1-8.*

Benvenuti, S., & Cardinali, A. (2018). The mental telescope: understanding the geometry of Euclid by learning the non-Euclidean one. In L. Gómez Chova, A. López Martínez, I. Candel Torres (Eds.), *Inted2018 Proceedings* (pp. 9296-9301). Valencia, Spain: IATED Academy.

Cardinali, A., & Benvenuti, S. (2020). Progress report of an ongoing project on theching non-Euclidean geometries at high school. In L. Gómez Chova, A. López Martínez, I. Candel Torres (Eds.), *EDULEARN20 Proceedings* (pp. 7555-7564). Valencia, Spain: IATED Academy. doi:10.21125/edulearn.2020.1917

Carroll, M., & Rykken, E. (2018). *Geometry: The Line and the Circle.* American Mathematical Society.

Haviger, J., & Vojkůvková, I., (2015). The van Hiele Levels at Czech Secondary Schools. *Procedia - Social and Behavioral Sciences, 171*, 912 – 918.

Hiele-Geldof, D. (1957). *The didactics of geometry in the lowest class of the secondary school* (Unpublished doctoral dissertation). University of Utrecht, Utrecht, Netherlands.

Hoffer, A. (1979). *Geometry (Teacher's Edition).* Menlo Park, CA: Addison-Wesley.

Hoffer, A. (1981). Geometry is more than proof. *Mathematics Teacher, 74*(1), 11-18.

Kline, M. (1972). *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press.

Magnani, L. (1978). *Le geometrie non euclidee. Testi di Euclide, Saccheri, Kant, Lobacevskij, Riemann, Beltrami, Helmholtz, Milhaud, Poincaré, Bachelard, Nagel* [Non-Euclidean geometries. Excerpts of Euclid, Saccheri, Kant, Lobacevskij, Riemann, Beltrami, Helmholtz, Milhaud, Poincaré, Bachelard, Nagel]. Zanichelli.

MIUR (1996). *Circolare ministeriale n. 615 del 27 settembre 1996 - Piano nazionale per l'introduzione dell'Informatica nelle scuole secondarie superiori. Indicazioni programmatiche relative all'insegnamento della matematica nel triennio del liceo ginnasio* [Ministerial Circular No. 615 of 27 September 1996 - National plan for the introduction of information technology in upper secondary schools. Programmatic guidelines on the teaching of Mathematics in the last three years of high school].

Toth, I. (2003). *No! Libertà e verità creazione e negazione. Palinsesto di parole e immagini* [No! Freedom and truth, creation and denial. Palimpsest of words and images]. Milano: Bompiani.

Usiskin, Z. (1969). *The Effects of Teaching Euclidean Geometry via Transformations on Student Attitudes and Achievement in Tenth Grade Geometry* (Unpublished doctoral dissertation), University of Michigan, USA.

Usiskin, Z. (1982). *Van Hiele Levels and Achievement in Secondary School Geometry*. University of Chicago. ERIC Document Reproduction Service.

Van Hiele, P. M. (1958-59). La signification des niveaux de pensee dans l'enseignement par la methode deductive [The significance of level of thought in teaching by the deductive method]. *Mathematica & Pedagogia, 16*.

Zheng, Y. (1992). Non-Euclidean geometry and revolutions in mathematics. In D. Gillies (Ed.), *Revolutions in Mathematics* (pp.169-182). Oxford University Press.

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